Generating multi-factor arbitrage-free scenario trees with global optimization

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Abstract

Simulation models of economic, financial and business risk factors are widely used to assess risk exposures and support decisions. Extensive literature on scenario generation methods aims at describing some underlying stochastic processes with the least number of scenarios to overcome the "curse of dimensionality". There is, however, an important issue that is usually overlooked when one departs from the application domain of security pricing: the no-arbitrage restriction. We formulate a moment matching model to generate multi-factor scenario trees satisfying no-arbitrage restrictions as a global optimization problem. While general in its formulation the resultant model is non-convex and can grow substantially even for a modest number of assets and scenarios. Exploiting the special structure of the problem we develop convex lower bounding techniques for its solution. Applications to some standard problems from the literature illustrate that this is a reliable approach to stochastic tree generation and is used to price a European basket option in complete and incomplete markets.

Keywords: scenario trees; global optimization; convex lower bounding; stochastic programming; pricing in incomplete markets.

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1 Introduction

Simulation models of economic, financial and business risk factors are widely used to assess risk exposures and support financial decision making, and risk management is often based on simulations of the risk factors of the balance sheet (see, e.g., Jamshidian and Zhu, 1997; Rebonato et al., 2005). Scenario trees are widely used in multistage stochastic programming, where the time dimension and non-anticipativity of future events are key features of the model (Carinõ and Ziemba, 1998; Consigli et al., 2010; Consiglio et al., 2006; Mulvey and Vladimirou, 1992). The recent trend is to address these issues at the enterprise-wide level, Dembo et al. (2000a), which broadens the risk factors to include not only financial but also economic and business. For a detailed review of literature on scenario methods for risk management and portfolio optimizaiton see Dupačová et al. (2000); Kaut and Wallace (2007) and (Zenios, 2007, chap. 9).

In synthesis, we identify three approaches:

1. The *moment matching approach* describes the joint distribution of scenarios in terms of moments including cross-moments to take into account inter-dependence of the factors. It solves a set of non-convex equations to match the mathematical expressions of the factor moments to exogenously given values. The main idea is found in the seminal paper by Høyland and Wallace (2001). Refinements suggested by Date et al. (2008); Høyland et al. (2003) reduce the computational complexity of the underlying optimization problem.

2. The *copula approach* postulates the distribution function of the marginals and then, by imposing an associative structure, determines the multivariate joint distribution. This method became popular as "copula approach" since the copula function is used to model dependencies among the variables (see Cherubini et al., 2004).

3. The *distance minimization approach* approximates the true distribution (continuous or discrete) with few mass points that minimize the (Kantorovich) distance between an original stochastic optimization model and the approximated one. The algorithmic implementation usually starts from set of points generated by a discrete reference process or by a discretization of a continuous process and then, using spe-

cific metrics, partitions the points in each stage into disjoint subsets that reduce the total number of scenarios and shape the tree structure (Dupačová et al., 2003; Hochreiter and Pflug, 2007).

(We leave out the extensive literature on simulations for security pricing, see, e.g. Glasserman (2004), which focuses on a specific problem and hence takes advantage of specific stochastic process structures; we take up this issue in the application section.)

A common aim of these methods is to approximate the underlying stochastic process or probability distribution with the least number of scenarios. However, an important issue is usually overlooked when one departs from the security pricing literature: the scenario approximation should not present arbitrage opportunities. The generation of arbitrage-free scenarios is complicated by the need to use two probability measures—the objective and the martingale—and returns compatible with both, to match the approximated process to the original.

The significance of no-arbitrage scenarios is of course well understood in the pricing literature. The problem has resurfaced in more complex forms in recent works where pricing options is done in the context of multi-period stochastic optimization models for portfolio management (Consiglio and De Giovanni, 2008; Topaloglou et al., 2008). But, even for portfolio optimization with simpler asset classes the absence of arbitrage is a key property. Geyer et al. (2010) has shown that scenario trees with arbitrage opportunities can produce spurious results when used in conventional portfolio models.

In a commentary to Høyland and Wallace (2001), Klaassen (2002) suggests two alternatives to handle arbitrage opportunities. One is to re-apply the scenario generation method from a different starting point, and/or increase the number of scenarios, in the hope that the newly generated set of scenarios is free of arbitrage. The other is to explicitly add no-arbitrage constraints to the original set of equations.

Adding a set of equations to generate arbitrage-free scenarios is a viable approach. However, it leads to a system of non-convex equations whose solution is prohibitive and to solve real-world applications (Høyland et al., 2003) proposed a heuristic that did not guarantee the arbitrage-free property (although it worked well for their applications).

Accordingly, arbitrage opportunities are usually eliminated by re-sampling

and/or increasing the number of scenarios. Such an approach is not free of faults or limitations. As shown in Geyer et al. (2012), increasing the number of scenarios does not surely yield arbitrage-free scenarios, as that depends on the structure of the expected returns and variance-covariance matrix. Moreover, increasing the number of scenarios could not be viable when the scenarios are used in multi-stage financial planning models, as the dimensionality of the model grows exponentially with the size of the tree.

This paper resolves the limitations of existing literature and in the process provides a very general methodology. We formulate the moment matching scenario generation model with no-arbitrage constraints as an optimization problem whose global minimal value is zero, if a solution exists (Maranas and Floudas, 1995). The resulting optimization problem is non-convex and local search algorithms can be trapped in local minima with non-zero value, thus leading to the erroneous conclusion that no solution exists. To overcome this difficulty we develop a global optimization approach based on convex lower bounding techniques (see Floudas and Gounaris, 2009, for a review) that takes advantage of the problem structure and is, therefore, computationally tractable.

Our paper makes two innovations: First, it formulates a global optimization model to generate moment-matching, arbitrage-free trees for an arbitrary number of risk factors (or, assets). Second, it develops an algorithm to exploit the special structure of the model, thus showing global optimization to be a robust tool for scenario generation. As a result of the model we obtain both *objective* (P) and *risk neutral* (Q) probability measures, and the developed methodology provides an extension of pricing models to distributions with general moments. In particular, since the *state price density* is obtained by the ratio Q/P for each node of the tree (Pliska (1997)) we can apply this method for pricing securities in incomplete markets, see Section 4.

The paper is organized as follows. In Sections 2 and 3 we formulate the model and develop the solution method. Section 4 reports on the implementation of this method to some standard models from the literature and to the pricing of a European basket option in complete and incomplete markets. Section 5 concludes.

2 Notation and model setup

We assume that asset returns follow stochastic processes in discrete space and time. The set of asset returns is labeled by index set $\mathcal{J} = \{1, 2, \ldots, J\}$ and are observed on a finite number of *time stages*, $t = 0, 1, 2, \ldots, T$:

$$R = \left(R_t^1, \ldots, R_t^J\right)_{t=0}^{T}. \tag{1}$$

(If some portfolio decision needs to be made at each stage t, such as in portfolio replication or portfolio optimization, these are called *decision stages*. Time stages for asset prices and decision stages for portfolios do not need to coincide but for simplicity we assume they do.)

The return process is modeled on the probability space $(\Omega, \mathcal{F}, P)$, where the sample space Ω is assumed to be finite. Such a formulation allows for a market representation through *scenario trees* (see Pliska, 1997). We denote by $\mathcal{N}_t$ the set of nodes at stage t. Each node $n \in \mathcal{N}_t$ corresponds, one–to–one, with an atom of the filtration $\mathcal{F}_t$. In the tree, every node $n \in \mathcal{N}_t$, $t = 1, \ldots, T$, has a unique ancestor node $a(n) \in \mathcal{N}_{t-1}$, and every node $n \in \mathcal{N}_t$, $t = 0, \ldots, T-1$, has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_t$. The collection of all the nodes is denoted by $\mathcal{N} \equiv \bigcup_{t=0}^{T} \mathcal{N}_t$.

In the probabilistic context, if we assume that the sample space Ω is finite, every algebra $\mathcal{F}$ corresponds to a partitioning of Ω into mutually disjoint subsets (the $\mathcal{F}$–atoms). In a scenario tree, there is a one-to-one map between the nodes $n \in \mathcal{N}_t$ and the partition sets $\mathcal{A}_t$, for each $t = 1, \ldots, T$. A filtration simply corresponds to a sequence of algebras generated by successively finer partitions of Ω (see Figure 1, left panel).

The tree displayed in the right panel of Figure 1 is general as the branching factor is allowed to vary in each stage. To simplify notation, we work with trees having the same number of child nodes per ancestor, $|\mathcal{C}(n)| = L$.

To form a tree for a given set of stages, we match the moments of the sub-tree emanating from each node, and repeat the matching procedure for each non-final node. Any temporal relationship between the moments, such as autocorrelation or GARCH effects, can be accounted exogenously by specifying the dynamics of the input data (Høyland and Wallace, 2001).

We describe now the equations and the variables for matching a generic sub-tree. L is the number of child nodes with ancestor $a(n)$ and $\mathcal{L} = \{1, 2, \ldots, L\}$ denotes the set of indices for the states of the economy (sce-

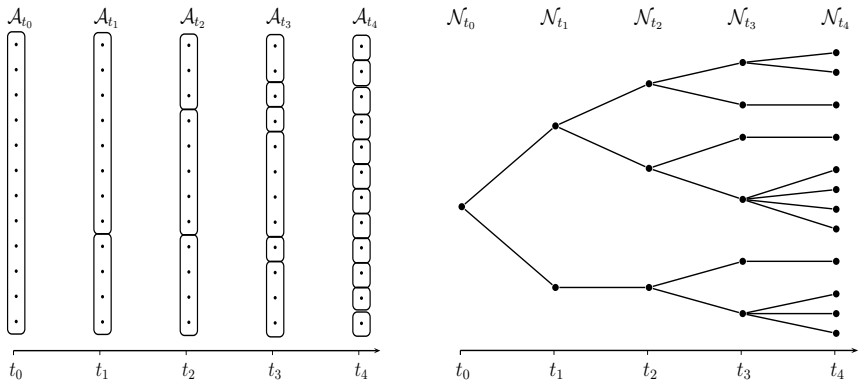

Figure 1: A finite filtration (left panel) and its associated tree (right panel).

narios) in the next period. Since we focus on matching sub-trees we drop the subscript t and let R_{jl} be the return of each asset $j \in \mathcal{J}$ and scenario $l \in \mathcal{L}$ for the generic sub-tree, and let p_l be the corresponding objective probability.

In practical applications we are usually interested in matching up to the first four central moments of the asset return distributions, and the correlations of each pair of assets. Let μ_j, σ_j, γ_j and κ_j denote, respectively, expected return, standard deviation, skewness and kurtosis, for each $j \in \mathcal{J}$, and ρ_{jh} the correlation of each pair $j, k \in \mathcal{J}$, and $j \neq k$.

The arbitrage-free moment matching problem is formulated as the system of nonlinear equations:

Problem 1. Arbitrage-free moment matching.

$$\sum_l p_l R_{jl} = \mu_j, \qquad j \in \mathcal{J} \tag{2}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)^2 = \sigma_j^2, \qquad j \in \mathcal{J} \tag{3}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)^3 = \gamma_j \, \sigma_j^3, \qquad j \in \mathcal{J} \tag{4}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)^4 = \kappa_j \, \sigma_j^4, \qquad j \in \mathcal{J} \tag{5}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)\left(R_{ki} - \mu_k\right) = \rho_{jk} \, \sigma_j \, \sigma_k, \qquad j,k \in \mathcal{J}, k > j \tag{6}$$

$$\sum_l q_l R_{jl} = r, \qquad j \in \mathcal{J} \tag{7}$$

$$\sum_l p_l = 1 \tag{8}$$

$$\sum_l q_l = 1 \tag{9}$$

$$q_l > 0, p_l \geq 0, \qquad l \in \mathcal{L}. \tag{10}$$

Problem 1 describes the matching of the moments and cross-moments of the joint probability distribution to exogenously given values μ_j, σ_j, γ_j, κ_j and ρ_{jk}. Eqns. (7) are the no-arbitrage constraints (Pliska, 1997), where, without loss of generality, we assume that r is the deterministic *risk free* rate. For a stochastic risk free rate, eqns. (7) are modified according to Pliska (1997) as

$$\sum_l q_l \frac{R_{jl} - r_l}{1 + r_l} = 0. \tag{11}$$

This model does not make any assumptions on the probability distributions of the returns of the asset classes. It simply matches observed moments consistently with the no-arbitrage theory. In this sense the model is quite general. This is an advantage for cases where no theoretical or empirical basis can justify any assumption on the underlying distributions, such as is the case for models that include both financial and economic random variables, or when business random variables are also included. When something more is known, this could be incorporated in the model through additional constraints. For instance, Cochrane and Saa-Requejo (2000) argue for "good

deal" bounds for incomplete markets and such considerations fit naturally in our model setup (although we can not vouch for the computational tractability of such extensions). If a distributional assumption can be made (e.g, lognormality) then one would need to estimate error bounds in addition to matching moments to assess if the calibrated tree is close to the assumed distribution. An important outcome of our approach is that the model may admit more than one arbitrage-free solutions and therefore it produces a range of plausible prices instead of a point estimate. We illustrate this point in the applications section.

3 A global optimization approach

We now develop a solution method for Problem 1 based on Maranas and Floudas (1995). They employ a partitioning strategy of the interval of the variables coupled with convex relaxations of the nonlinear terms of each equation, and we specialize this approach to exploit the structure of Problem 1.

3.1 Variable bounds and scaling

First, we standardize the variable of the problem. In particular, denote by

$$z_{jl} = \frac{R_{jl} - \mu_j}{\sigma_j} \tag{12}$$

the standardized returns R_{jl} to obtain $R_{jl} = \mu_j + \sigma_j \, z_{jl}$, and substitute in equations (2)–(10).

Second, we define proper bounds for each variable. This is important as the solution search proceeds through successively finer partitions of the hyper-rectangle specified by the variable bounds, and the smaller this initial range the faster the convergence. Natural bounds are available for the variables p_l and q_l: since they are probabilities $p_l, q_l \in (0, 1]$ [1]. Less obvious are the bounds on z_{jl} since $z_{jl} \in (-\infty, \infty)$. However, as z_{jl} denotes standardized returns we set $R_{jl} > -1$ to rule out negative prices. Therefore the lower bound for the standardized variable is

$$\underline{z}_j = \frac{-1 - \mu_j}{\sigma_j}. \tag{13}$$

[1] According to theory, risk neutral probabilities have to be strictly greater than zero.

Further restrictions of the range of z_{jl} can be imposed by analysis of historical price series. In general we bound the variable to stay within 3 to 5 standard deviations from the mean, with larger bounds being appropriate for higher kurtosis, although the efficiency of the algorithm deteriorates.

With this transformations we can now develop our solution approach using convex relaxation of *posynomial* functions (see Appendix A for a formal definition). Such functions are characterized by strictly positive variables. The positivity of p_l and q_l is rooted in their meaning as probabilities but for z_{jl} a suitable transformation is needed. Furthermore, to enhance numerical stability of the algorithm we scale all variables to have the same range, and in particular the range of p_l and q_l.

The transformation of z_{jl} is given by:

$$\begin{aligned} z_{jl} &= \underline{z}_j + t_{jl}(\overline{z}_j - \underline{z}_j) \\ &= \underline{z}_j + t_{jl}\Delta_j, \end{aligned} \tag{14}$$

where $0 < \underline{t}_j \leq t_{jl} \leq \bar{t}_j = 1$ and $\Delta_j = \overline{z}_j - \underline{z}_j$. Substituting eqn. (14) in the standardized eqns. (2)–(7), and after some algebra, we obtain:

Problem 2. Arbitrage-free moment matching with scaled variables.

$$\sum_l p_l\, t_{jl} = A_j, \qquad j \in \mathcal{J} \tag{15}$$

$$\sum_l p_l\, t_{jl}^2 = B_j, \qquad j \in \mathcal{J} \tag{16}$$

$$\sum_l p_l\, t_{jl}^3 = C_j, \qquad j \in \mathcal{J} \tag{17}$$

$$\sum_l p_l\, t_{jl}^4 = D_j, \qquad j \in \mathcal{J} \tag{18}$$

$$\sum_l p_l\, t_{jl}\, t_{kl} = F_{jk}, \qquad j,k \in \mathcal{J}, k > j \tag{19}$$

$$\sum_l q_l\, t_{jl} = H_j, \qquad j \in \mathcal{J} \tag{20}$$

$$\sum_l p_l = 1, \tag{21}$$

$$\sum_l q_l = 1, \tag{22}$$

$$0 < \underline{p}_l \leq p_l \leq \overline{p}_l = 1, \qquad l \in \mathcal{L} \tag{23}$$

$$0 < \underline{q}_l \leq q_l \leq \overline{q}_l = 1, \qquad l \in \mathcal{L} \tag{24}$$

$$0 < \underline{t}_j \leq t_{jl} \leq \overline{t}_j = 1, \qquad j \in \mathcal{J}, l \in \mathcal{L}. \tag{25}$$

The derivations of the transformed equations with scaled variables are reported in Appendix B.

3.2 The branch & bound algorithm

Finding all the solutions of Problem 2 is now re-formulated as a global optimization problem. Following Maranas and Floudas (1995), we index by $m \in \mathcal{M}$ the equations of the model, i.e., $\mathcal{M} = \{1, 2, \ldots, M\}$ is the index set of eqns. (15)–(20). We denote by $\boldsymbol{\beta}$ a vector stacking the variables p_l, q_l, t_{jl}, and by $\underline{\boldsymbol{\beta}}$, $\overline{\boldsymbol{\beta}}$, respectively, their lower and upper bounds. We also denote by $e_m(\boldsymbol{\beta})$ the difference between the value of the equation at $\boldsymbol{\beta}$ and its right-hand-side term, for each $m \in \mathcal{M}$.

Let s be a scalar slack variable. Then the following inequality constrained problem amounts to solving the system of equations (15)–(25):

Problem 3. Inequality-constrained minimization

$$\min_{\boldsymbol{\beta},s} s \tag{26}$$

$$\textbf{s.t.} \tag{27}$$

$$e_m(\boldsymbol{\beta}) - s \leq 0, \qquad m \in \mathcal{M} \tag{28}$$

$$-e_m(\boldsymbol{\beta}) - s \leq 0, \qquad m \in \mathcal{M} \tag{29}$$

$$\underline{\boldsymbol{\beta}} \leq \boldsymbol{\beta} \leq \overline{\boldsymbol{\beta}}. \tag{30}$$

An optimal solution $(\boldsymbol{\beta}^*, s^*)$ of Problem 3 with $s^* = 0$ is a solution of the system (15)–(25). In turn, a global optimum with a non-zero s^* denotes an infeasible system of equations. Note that, since the equations involved are, in general, non-convex, a local optimization algorithm could lead to solutions which are locally optimal thus missing the global optima. Even worse, if a local minimum has a non-zero objective value we (erroneously) conclude that no feasible solutions exist to the original system. Hence we need a solution method that can identify all solutions. In practice we may terminate once a zero solution is found.

Global optimization algorithms to solve non-convex problems have been widely studied. They are mainly subdivided in three classes: deterministic, stochastic and meta-heuristic. Deterministic global optimization algorithms are usually based on a branch & bound search strategy, where the *bound phase* is implemented by minimizing a *convex relaxation* of Problem 3. On each sub-rectangle $[\underline{\boldsymbol{\beta}'}, \overline{\boldsymbol{\beta}'}] \subset [\underline{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}}]$, obtained in the *branching phase*, the constrained global minimum of the convex relaxed problem can be routinely found with any local optimization algorithm. Note that, since the convex relaxation is obtained by a convex underestimation of each non-convex term of the model equations, the minimum of the relaxed problem will be an underestimation of the global minimum. This implies that if the relaxed global minimum is positive, then the relative partition can be fathomed, as the slack variable s cannot be driven to zero, and therefore the moment matching model has no solution in that specific partition. On the other hand, if the relaxed global minimum is negative, then no conclusion can be drawn and the interval is further partitioned. The algorithm terminates when all the hyper-rectangles with a negative lower bound cannot be further

partitioned, or, in practice, when their norm is within a given tolerance: $\|\overline{\boldsymbol{\beta}'} - \underline{\boldsymbol{\beta}'}\| \leq \epsilon_d$.

In Appendix A we give a convex reformulation of the model that exploits its special structure and allows for efficient solutions.

4 Applications

In this section we apply the method to generate scenarios for some problems from the literature and to price a basket option in complete and incomplete markets. We carry out experiments to assess the performance of our approach, compare to available software for scenario generation and assess the quality of the scenarios obtained when used for security pricing.

The data sets are taken from real problem instances and they contain asset classes ranging from cash to stock. In particular, we perform the experiments on data sets from Høyland et al. (2003) and the FINLIB library of Consiglio et al. (2009), and label them, respectively, `HKW-X` and `FINLIB-Y`, where `X=8,12,20` and `Y=15,20` denote the number of assets in each data set.

All the experiments are carried out on a Linux machine with 2.00 GHz Xeon Quad-Core. The convexified problems are solved with GAMS/CONOPT or GAMS/SNOPT.

4.1 Checking for arbitrage

As noted earlier, neither re-sampling nor increasing the number of scenarios are foolproof approaches to generate arbitrage-free scenarios. Table 1 summarizes results with the generation of no-arbitrage scenarios using the heuristic of Høyland et al. (2003). For a given number of scenarios, we re-sample 100 different instances and assess the presence of arbitrage by solving the stochastic programming model of King (2002). An unbounded solution signals arbitrage. The table summarizes the success rate of producing arbitrage-free scenarios by simple re-sampling and we note that the percentage of no-arbitrage scenarios depends on the test set, i.e., on the structure of the variance-covariance matrix (see Geyer et al. (2012)).

Increasing the number of scenarios, as suggested by Klaassen, improves the success rate (although, for problem `HKW_20` it was not possible to generate arbitrage-free scenarios). The success rate is lower for problems with more assets and the number of scenarios needed are on average more than double

the number of assets. This is crucial for practical applications. As we will see in the next section, to price an option in complete or incomplete markets, we need to build scenario trees, that grow exponentially with the number of time steps and scenarios. Therefore, a desirable property of arbitrage-free scenarios is to match the moments of the distribution with the minimum number of scenarios. According to theory (see Pliska, 1997), the number of scenarios should be equal to the number of assets plus one, $|\mathcal{L}| = |\mathcal{J}| + 1$. Note that if it is possible to generate trees such that each sub-tree has a number of scenarios equal to $|\mathcal{J}|+1$, then the option price can be determined by simply discounting the final payoff under the risk neutral measure, given in our model by the probabilities q_l, for $l \in \mathcal{L}$.

	Number of Scenarios					
Problem	10	15	20	30	50	100
`HKW-8`	0%	6%	25%	63%	93%	100%
`HKW-12`	NA	0%	0%	11%	56%	100%
`HKW-20`	NA	NA	NA	0%	0%	0%
`FINLIB-15`	NA	NA	0%	0%	0%	12%
`FINLIB-20`	NA	NA	NA	0%	0%	6%

Table 1: Percentage of no-arbitrage scenarios for each test set and for different number of scenarios. *NA* indicates that absence of arbitrage cannot hold because the number of scenarios does not exceed the number of assets.

4.2 Accuracy of the solution

In this section we show that the global optimization approach is a feasible alternative to re-sampling procedures, that, as seen above, perform poorly in terms of success rate and in terms of number of scenarios needed to guarantee absence of arbitrage.

Our objective is to generate sets of scenarios with the minimum number of branches, possibly satisfying the *completeness hypothesis* whereby the number of scenarios equals the number of assets plus one. We point out that the global optimization approach is able to locate all the global minima (scenario trees) of the problem. If the system of equations is consistent, the moment matching problem could have infinite solutions and in our experiments we terminate after ten solutions.

In Table 2 we display the average maximum error for each set of equations of the moment matching problem. That is, for each set of equations

corresponding to the moment to be matched, we record the maximum error obtained over the set of assets, where error is the difference between the value of the expression on the left-hand-side of the equation and the parameter on the right-hand-side[2]. This value is then averaged over the ten generated trees. For example, the column denoted by ρ displays the maximum mismatch over all the equations describing the cross-correlations, averaged over the ten solutions (scenario trees) . Observe that the average maximum error is quite negligible, meaning that the solutions found match very closely the empirical moments for all test problems. Moreover, the trees generated have the minimum number of scenarios required to exclude arbitrage opportunities.

Problem	μ	σ	γ	κ	ρ	r
HKW-8	1.08E-06	4.20E-06	2.84E-05	1.40E-04	4.88E-06	3.87E-07
HKW-12	1.38E-06	9.14E-06	6.45E-05	2.51E-04	5.71E-06	2.70E-07
HKW-20	7.22E-06	8.91E-06	7.86E-05	3.34E-04	1.50E-05	2.60E-07
FINLIB-15	1.41E-06	3.83E-06	5.26E-05	2.49E-04	5.56E-06	1.12E-07
FINLIB-20	2.13E-06	7.38E-06	6.00E-05	1.58E-04	8.03E-06	3.66E-07

Table 2: Average maximum error for each moment matched.

To confirm robustness of the solution algorithm, we display in Table 3 the standard deviations of the maximum error over the ten different trees.

Problem	μ	σ	γ	κ	ρ	r
HKW-8	8.67E-07	3.67E-06	2.34E-05	1.30E-04	7.83E-06	5.89E-07
HKW-12	1.52E-06	8.11E-06	3.06E-05	1.70E-04	9.43E-06	3.94E-07
HKW-20	9.69E-06	6.62E-06	1.71E-05	1.07E-04	1.40E-05	2.43E-07
FINLIB-15	2.49E-06	5.38E-06	3.38E-05	2.06E-04	7.04E-06	1.34E-07
FINLIB-20	2.46E-06	5.40E-06	3.37E-05	1.13E-04	6.76E-06	7.64E-07

Table 3: Standard deviation of the maximum error of ten different trees for each moment matched.

Finally, we report in Table 4 the computational times to find one and ten solutions. Solution times increase with the number of assets, primarily due to the exponential nature of the branch & bound algorithm.

[2]The symbols of the columns are those used to describe the equations of Problem 1. For readability, we omit the errors for the normalization constraints for p_l and q_l, which are in the range 3.26E-6 to 9.33E-7.

	T_1	T_{10}
HKW-8	0:00:09	0:01:41
HKW-12	0:02:00	0:15:02
HKW-20	0:43:41	4:50:14
FINLIB-15	0:00:53	1:08:57
FINLIB-20	0:04:58	2:51:31

Table 4: Solution times for one (T_1) and ten (T_{10}) solutions in hrs:min:sec.

4.3 Options pricing applications

We showed that it is possible to built trees to match with high accuracy a given set of moments and obtain the corresponding objective and risk neutral measures. As we pointed out this problem has extensive applications in risk management, especially when using multiperiod optimization models. Here we illustrate two applications of this method with options pricing in complete and incomplete markets. We will see that high-quality solutions are obtained even with the very low levels of granularity we consider.

4.3.1 Complete markets

We start by assessing the quality of our trees with respect to a financial problem whose solution, under some assumptions, can be obtained with other methods. We consider the pricing of a European basket option, written on $|\mathcal{J}| = 4$ equally weighted assets, maturity $T = 5$ years and correlation among the underlying assets 0.5. The objective of such an experiment is to assess the quality of our scenario trees. For real-world evaluations of basket options under normality assumptions more efficient methods are available; things become more complex if we need to take into account stylized facts, such as fat-tails or skewness of the distributions of the underlying assets. We compute the price of such an option assuming normality using standard Montecarlo simulation with 1,000,000 scenarios drawn from the gaussian risk-neutral distribution.

For our method, the Black-Scholes (BS) hypotheses is satisfied by generating scenarios with drift $\mu = r$, (without loss of generality we set the risk free $r = 0$), standard deviations $\sigma = 10\%, 20\%, 30\%$ per year, skewness $\gamma = 0$, kurtosis $\kappa = 3$, and correlation among the four assets $\rho = 0.5$. Note that $\gamma = 0$ and $\kappa = 3$ is a requirement to satisfy the gaussian hypothesis of the BS model. Matching only these two moments does not ensure that

the distributions of the asset returns is normal (in theory, infinite moments of the gaussian should be matched). For market completeness we set the number of scenarios $|\mathcal{L}| = |\mathcal{J}| + 1$.

The 5-year tree is constructed sequentially, that is, at each branching node a 1-year subtree is generated by means of the optimization procedure described in the previous sections. For consistency with BS we assume that the input moments of the conditional distributions are time independent. Therefore, the 5-year tree is obtained by generating a single 1-year subtree and replicating it to each branching node.

Following the notation in Section 2, we denote by $\mathcal{N}_T$ the set of nodes at the final period that coincides with the maturity of the option. There is a unique path from the root node to the final nodes, and we denote by $\mathcal{H}(n)$ the index set of nodes which belongs to the path leading from the root of the tree to the final nodes $n \in \mathcal{N}_T$.

For each final node n, we compute the price of each asset $j \in \mathcal{J}$ as follows:

$$P_n^j = P_0^j \prod_{m \in \mathcal{H}(n)} \left(1 + R_m^j\right), \tag{31}$$

where, R_m^j is the return of the asset j at node m (with $R_0^j = 0$), and P_0^j is the price of the asset j at the root node $m = 0$ (we set $P_0^j = 100$, for each $j \in \mathcal{J}$). We compute in a similar way the risk neutral probabilities attached to each final node n, i.e.,

$$q_n^* = \prod_{m \in \mathcal{H}(n)} q_m, \tag{32}$$

where, q_m is the risk neutral probability of each node $m \in \mathcal{H}(n)$, and $q_0 = 1$.

The price of the option is now obtained as the present value of the expected value of the final payoff under risk-neutral probabilities q_n^*, $n \in \mathcal{N}_T$:

$$C = e^{-r} \sum_{n \in \mathcal{N}_T} q_n^* \max(P_n^B - K, 0), \tag{33}$$

where $P_n^B = \sum_{j \in \mathcal{J}} w_j P_n^j$. We remark that due to the imposed market completeness it is possible to price the option by simple discounting of expected value under q^*.

We fit scenario trees for different volatilities $\sigma = 10\%, 20\%, 30\%$ and price the basket option for strike prices ranging from 80 to 120 (the ini-

tial price of each asset $P_0^j = 100$). We generate twenty solutions of the non-convex optimization problem, and also use Montecarlo simulation with 1,000,000 scenarios to estimate the "true" price under the normality assumption. The results are illustrated in Figure 2 where we show the range of prices obtained by the calibrated trees and the Montecarlo estimate. In all cases –except for deep in-the-money and deep out-of-the-money options under the extreme volatility scenarios– the range of prices obtained from the scenario tree brackets the Montecarlo estimate and the range is small.

In Table 5 we compute the average price over the twenty trees obtained by eqn. (33), and report the mean absolute percentage difference between the average price from the trees and the Montecarlo price estimate.

	$\sigma = 10\%$			$\sigma = 20\%$			$\sigma = 30\%$		
	MC	Opt.	Error	MC	Opt.	Error	MC	Opt.	Error
80	20.80	20.81	0.49%	25.10	25.17	1.69%	30.59	30.36	0.76%
85	16.60	16.74	1.04%	21.89	21.84	0.97%	27.90	27.77	0.79%
90	12.87	12.98	1.38%	18.99	18.89	0.89%	25.43	25.51	1.74%
95	9.68	9.72	0.91%	16.40	16.43	1.93%	23.17	23.45	2.79%
100	7.06	7.21	2.16%	14.10	14.20	3.06%	21.10	21.45	3.17%
105	5.00	5.12	2.83%	12.08	12.15	3.62%	19.21	19.49	2.75%
110	3.45	3.37	3.40%	10.32	10.29	2.81%	17.49	17.55	1.46%
115	2.32	2.15	7.99%	8.79	8.59	2.24%	15.92	15.62	1.84%
120	1.52	1.43	7.50%	7.46	7.07	5.61%	14.49	13.73	5.29%

Table 5: Montecarlo price MC of the basket option obtained by drawing 1,000,000 scenarios from the risk neutral distribution, average price "Opt." using twenty 5-year trees and mean absolute error between the two values.

Note that, for in-the-money and at-the-money options (strike prices ≤ 105) the option price error is fairly small. The quality of the solution deteriorates somewhat for out-of-the-money and low volatility. In reality, on average the option price is fairly close to the Montecarlo price. However, the coarse granularity of the tree (only $5^5 = 3125$ scenarios versus 1,000,000 of the Montecarlo) yields option prices very close to zero in some tree instances.

We point out that the aim of this experiment is to establish the quality of the generated trees and point out that there are more accurate methods to price basket options under the BS hypotheses. The value of this method is not so much the accuracy of the average price it computes vis-a-vis the Montecarlo price, but the fact that it generates a range of prices that

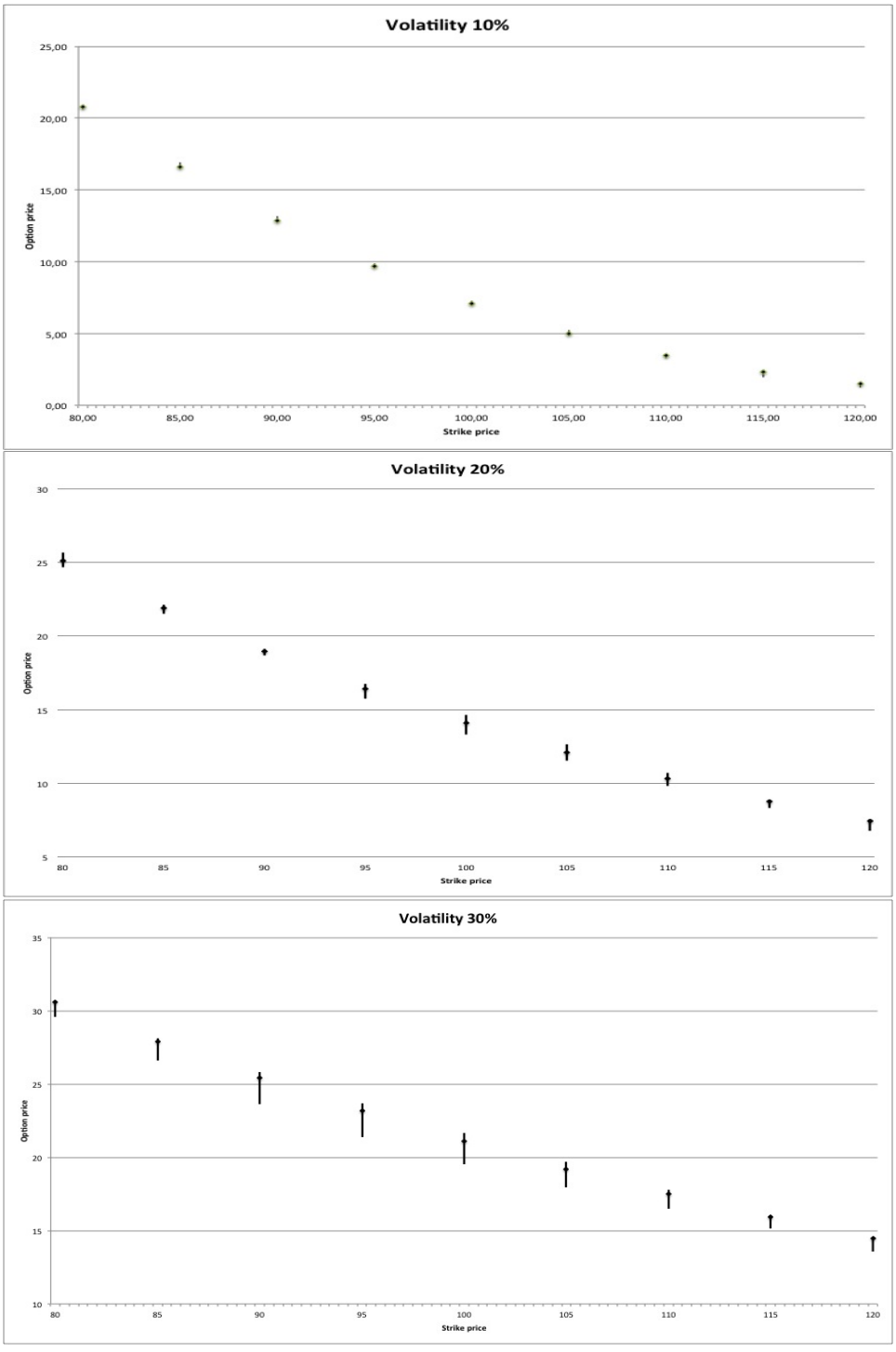

Figure 2: The range of prices obtained from twenty calibrated trees for different strike prices and volatilities; the Montecarlo estimate under normality assumptions is indicated by the bullet point.

bracket the Montecarlo price. That is, if normality holds then our method approximates accurately the Montecarlo estimate even with very few (3125) scenarios. But in the absence of any distributional information and with only the empirically observed moments to go by, the range of prices is a better indicator. Furthermore, the methodology can be applied to pricing more complex instruments, path-dependent options and in incomplete markets, which we consider next.

4.3.2 Incomplete markets

The representation through trees of the underlying stochastic process is particularly suitable to price option in incomplete markets, see, e.g., Dembo et al. (2000b). Market incompleteness arises when the number of risky factors is greater than the available securities to hedge them. This is simulated in our experiments by assuming a non-traded underlying asset.

In case of incompleteness, the martingale measures are infinite, and, therefore, there are infinite prices of the option under scrutiny, lying between a lower (bid price) and an upper (ask price) bound. Such bounds can be determined by appropriately modeling the hedging process. In our experiment we adopt a super-replication strategy, where at each node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$, the portfolio of assets is self-financing, and at each final node, $n \in \mathcal{N}_T$, the hedging portfolio super-replicates the option payoff. Such a strategy is equivalent to solving a linear stochastic programming model (one for the buyer and one for the writer), where the value of the portfolio at the root node is the option price (King, 2002). In this context a reference value to serve as the "true" value of the option is not available, and we assess the quality of the scenario trees through sensitivity analysis of the option price.

In Figure 3 we display the ask and bid prices of the basket option, with volatility $\sigma = 20\%$, and strike prices ranging from 80 to 120. The incomplete market prices, over the twenty trees, is obtained by assuming that one of the assets cannot be traded. More detailed statistics are reported in Table 6.

We observe that also in incomplete markets the trees are close to each other in the sense that they yield very similar option prices, as evidenced by the low standard deviation over the sample of twenty trees.

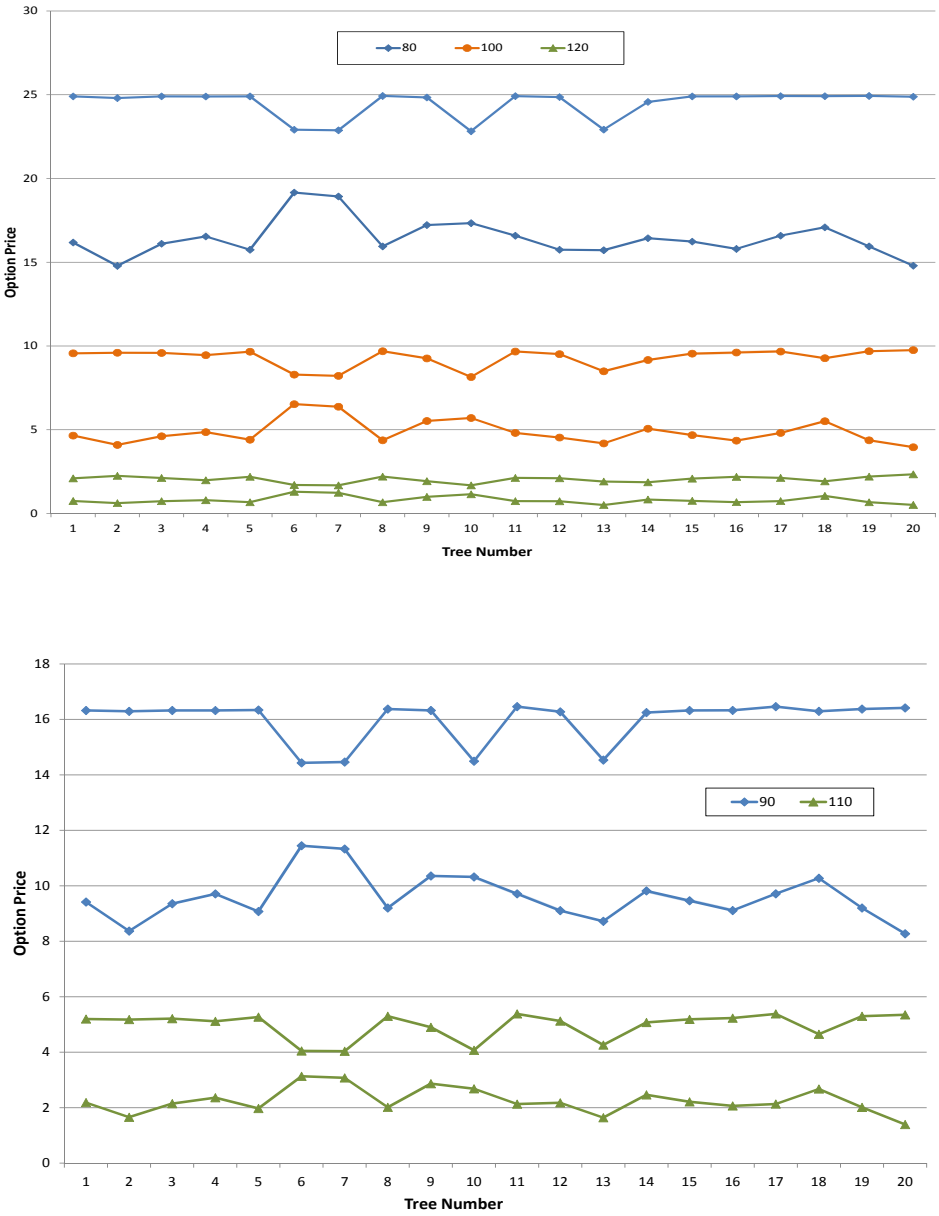

Figure 3: Bid/ask prices of the basket option in an incomplete market for different strike prices and trees, and volatility of assets $\sigma = 20\%$

	Buyer Side				
Strike Price	80	90	100	110	120
Mean	16.44	9.60	4.87	2.25	0.80
St. dev.	1.085	0.815	0.698	0.447	0.218
Min	14.78	8.27	3.95	1.39	0.50
Max	19.16	11.45	6.53	3.13	1.30
	Writer Side				
Strike Price	80	90	100	110	120
Mean	24.48	15.97	9.29	4.96	2.03
St. dev.	0.800	0.746	0.528	0.461	0.189
Min	22.82	14.43	8.15	4.04	1.67
Max	24.93	16.46	9.75	5.38	2.34

Table 6: Descriptive statistics of the price of a basket option in incomplete markets over a sample of twenty trees.

5 Conclusions

The generation of arbitrage-free scenario trees that match the moments of a set of risk factors is a prevalent problem in risk management and pricing financial instruments, and especially so in incomplete markets and for enterprise-wide risk management. We proposed a model formulation that casts this problem as a global optimization model whose solution is zero if a solution to the original problem exists. Exploiting the special structure of the model with linear relaxations of a convex reformulation, we have shown that the method is robust and computationally tractable. Experiments highlighted the efficacy of the methodology in pricing synthetic options in complete and incomplete markets. The result is a general purpose methodology that can generate theoretically correct and accurate scenarios with no more input requirements than the available moments.

A A convex reformulation of the model

Once Problem 2 is transformed into the equivalent constrained minimization Problem 3, we are faced with a set non-linear inequalities of the general form

$$\sum_k \prod_i x_{ki}^{\alpha_{ki}} - M - s \leq 0 \tag{34}$$

$$-\sum_k \prod_i x_{ki}^{\alpha_{ki}} + M - s \leq 0, \tag{35}$$

where, each term of the summation is a *posynomial*. In particular, a posynomial function is defined as

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}, \tag{36}$$

where $\alpha_i \in \mathbb{R}$, $c > 0$, and $\underline{x}_i > 0 \leq x_i \leq \overline{x}_i$, for each $i = 1, 2, \ldots, m$.

Observe that the nonlinear terms in eqns. (15)–(25) are of posynomial functional form, and this is exploited in the convex relaxation phase of the solution algorithm. Convexification of posynomial functions is carried out through the variable transformation $x \to f(y)$, where $f(y) : \mathbb{R} \to \mathbb{R}$ is a suitable mapping carrying the one-to-one relation between the original variable x and the transformed variable y. For example, Maranas and Floudas (1997) use an exponential transformation, $x \to e^y$; Tsai and Lin (2007) adopt a reciprocal transformation, $x \to y^{-1}$, which is a special case of the power transformation $x \to y^\beta$. Note that, in general, not all variables need to be transformed. For example, eqns. (21) and (22) are linear in p and q.

The inverse nonlinear transformation $y \to f^{-1}(x)$ has to be included into the transformed problem, thus moving the non-convexities from the original constraints to the inverse equality constraints. (For example, by using the power transformation, the non-convex definitional equality $y = x^{1/\beta}$ has to be associated with each transformed variable). In this respect, Lundell et al. (2009) approximate the inverse linear transformation constraint through a piecewise linear function, thus turning the model to a mixed-integer nonlinear program, while Lu et al. (2010) adopt an *ad-hoc* linear relaxation of the bilinear equation associated to each transformed variable. Since tree generation problems are characterized by a medium to high level of dimensionality, we adopt the approach of Lu et al. that does not require integer

variables that complicate the model. (For example, with number of assets $|\mathcal{J}| = 4$, and number of scenarios $|\mathcal{L}| = 5$, we obtain a nonlinear program with 30 variables and 37 constraints. In general, the number of variables are $2 \cdot |\mathcal{L}| + |\mathcal{J}| \cdot |\mathcal{L}|$, and the number of equations (excluding the box constraints) $2 + 5 \cdot |\mathcal{J}| + (|\mathcal{J}|^2 - |\mathcal{J}|)/2$.)

The starting point of the analysis in Lu et al. is based on the following proposition:

Proposition 1. *A twice differentiable function*

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i} \tag{37}$$

is convex if $\alpha_i < 0$ *and* $x_i > 0$*, for each* $i = 1, 2, \ldots, n$*.*

Let $\mathcal{K} = \{i | \, \alpha_i < 0, i = 1, 2, \ldots, n\}$ and $\overline{\mathcal{K}} = \{i | \, \alpha_i > 0, i = 1, 2, \ldots, n\}$. A convex reformulation of the posynomial function (36) is given by

$$f\left(\{x_i\}_{i \in \mathcal{K}}, \{y_i\}_{i \in \overline{\mathcal{K}}}\right) = \prod_{i \in \mathcal{K}} x_i^{\alpha_i} \prod_{i \in \overline{\mathcal{K}}} y_i^{-\frac{\alpha_i}{\delta_i}}, \tag{38}$$

where, $y_i = x_i^{-\delta_i}$ and $0 < \delta_i \leq 1$, for each $i \in \overline{\mathcal{K}}$.

For those variables x_i, $i \in \overline{\mathcal{K}}$, that appears in non-convex terms, it is necessary to relax the definitional equation $y_i = x_i^{-\delta_i}$. Lu et al. show that such relaxation is given by the following linear inequality, for each $i \in \overline{\mathcal{K}}$:

$$y_i \leq \underline{x}_i^{-\delta_i} + \frac{\overline{x}_i^{-\delta_i} - \underline{x}_i^{-\delta_i}}{\overline{x}_i - \underline{x}_i} \left(x_i - \underline{x}_i\right). \tag{39}$$

The parameters δ_i play an important role in the relaxation. In particular, the smaller the δ_i the tighter the convex relaxation. However, δ_i cannot be chosen arbitrarily close to zero and it has to be determined as a function of the computer accuracy and bounds $\overline{x}_i$, $\underline{x}_i$; see Lu et al. (2010), section 3.

To convexify Problem 2, we need to transform each variable p_l, q_l and t_{jl} since their exponents are all positive. Accordingly, we set:

$$\pi_l = p_l^{-\xi_l} \quad \text{and} \quad 0 < \xi_l \leq 1, \tag{40}$$

$$\chi_l = q_l^{-\upsilon_l} \quad \text{and} \quad 0 < \upsilon_l \leq 1, \tag{41}$$

$$\tau_{jl} = t_{jl}^{-\delta_{jl}} \quad \text{and} \quad 0 < \delta_{jl} \leq 1. \tag{42}$$

Remark 1. The variables p_l and q_l are the only ones to be relaxed, and this is because such variables appear in the linear eqns. (21)–(22). So we have to add to the convexified problem the following inequalities, for each $l \in \mathcal{L}$:

$$\pi_l \leq \underline{p}_l^{-\xi_l} + \frac{\overline{p}_l^{-\xi_l} - \underline{p}_l^{-\xi_l}}{\overline{p}_l - \underline{p}_l} \left(p_l - \underline{p}_l \right), \tag{43}$$

$$\chi_l \leq \underline{q}_l^{-\upsilon_l} + \frac{\overline{q}_l^{-\upsilon_l} - \underline{q}_l^{-\upsilon_l}}{\overline{q}_l - \underline{q}_l} \left(q_l - \underline{q}_l \right), \tag{44}$$

Remark 2. The inequality constraints of Problem 3 are of two types: (28) is the sum of positive terms, also known as *posynomial*; (29) is the sum of negative terms, also know as *signomial*. As we have seen, the transformation $y_i = x_i^{-\delta_i}$ turns the posynomials to a convex function. Similarly, a signomial will be turned to a concave function. We can relax each concave term by underestimating it through an affine function $T(y_1, y_2, \ldots, y_n)$. For example, the concave inequalities correspondent to the transformation of eqn. (15) are given by,

$$-\sum_l \pi_l^{-1/\xi_l} \tau_{jl}^{1/\delta_{jl}} + A_j - s \leq 0, \quad \text{for each } j \in \mathcal{J}. \tag{45}$$

Each concave term $G_{jl}(\pi_l, \tau_{jl}) = -\pi_l^{-1/\xi_l} \tau_{jl}^{1/\delta_{jl}}$ is underestimated by means of the affine function $T_{jl}(\pi_l, \tau_{jl}) = a_{jl}\, \pi_l + b_{jl}\, \tau_{jl} + c_{jl}$. The coefficients a_{jl}, b_{jl}, and c_{jl} are chosen in such a way that the affine function, $T_{jl}(\pi_l, \tau_{jl})$, is equal to the concave function, $G_{jl}(\pi_l, \tau_{jl})$, at the vertex of the rectangular domain identified by the upper and lower bounds of the variables π_l and τ_{jl}.

B The model with scaled variables

The derivation of eqns. (15)–(20) needs some tedious algebra especially for higher moments. We derive here only eqns. (15)–(16) and for the remaining parameters we provide the relations without the derivation, which follows along the lines described here. By substituting the scaling relation for z_{jl} in

the standardized eqn. (2), we obtain:

$$\sum_l p_l(\underline{z}_j + t_{jl}\Delta_j) = \underline{z}_j + \Delta_j \sum_l p_l t_{jl} = 0.$$

By isolating the summation of the last relation, we obtain:

$$\sum_l p_l t_{jl} = -\frac{\underline{z}_j}{\Delta_j} = A_j.$$

We proceed similarly for the standardized eqn. (3) to obtain:

$$\begin{aligned}
&\sum_l p_l(\underline{z}_j + t_{jl}\Delta_j)^2 = \\
&\sum_l p_l \left[\underline{z}_j^2 + t_{jl}^2\Delta_j^2 + 2\underline{z}_j t_{jl}\Delta_j\right] = \\
&\underline{z}_j^2 + \Delta_j^2 \sum_l p_l t_{jl}^2 + 2\underline{z}_j\,\Delta_j \sum_l p_l t_{jl} = \\
&\underline{z}_j^2 + \Delta_j^2 \sum_l p_l t_{jl}^2 + 2\underline{z}_j\,\Delta_j \left(-\frac{\underline{z}_j}{\Delta_j}\right) = \\
&\underline{z}_j^2 + \Delta_j^2 \sum_l p_l t_{jl}^2 - 2\underline{z}_j^2 = \\
&\Delta_z^2 \sum_l p_l t_{jl}^2 - \underline{z}_j^2 = 1.
\end{aligned}$$

Isolating the summation of the last relation, we obtain:

$$\sum_l p_l t_{jl}^2 = \frac{1 + \underline{z}_j^2}{\Delta_j^2} = B_j.$$

It is possible to verify that:

$$C_j = \frac{\gamma_j - \underline{z}_j\left[\underline{z}_j^2 + 3\right]}{\Delta_j^3}, \qquad D_j = \frac{\kappa_j - \underline{z}_j\left[4\gamma_j - \underline{z}_j^3 - 6\underline{z}_j\right]}{\Delta_j^4},$$

$$F_{jh} = \frac{\rho_{jh} + \underline{z}_j\,\underline{z}_h}{\Delta_j\,\Delta_h}, \qquad H_j = \frac{r - \mu_j - \underline{z}_j\,\sigma_j}{\sigma_j\,\Delta_j}.$$

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