

Designing guarantee options in defined contributions pension plans

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Abstract

The shift from defined benefits (DB) to defined contributions (DC) is pervasive among pension funds, due to demographic changes and macroeconomic pressures. In DB all risks are borne by the provider, while in plain vanilla DC all risks are borne by the beneficiary. For DC to provide income security some kind of guarantee is required. A minimum guarantee clause can be modeled as a put option written on some underlying *reference portfolio* of assets and we propose a discrete model that optimally selects the reference portfolio to minimise the cost of a guarantee. While the relation DB-DC is typically viewed as a binary one, the model setup can be used to price a wide range of guarantees creating a continuum between DB and DC. Integrating guarantee pricing with asset allocation decision is useful to both pension fund managers and regulators. The former are given a yardstick to assess if a given asset portfolio is fit-for-purpose; the latter can assess differences of specific reference funds with respect to the optimal one, signalling possible cases of moral hazard. We develop the model and report extensive numerical results to illustrate its potential uses.

Keywords: Pensions, minimum guarantee, defined benefits, defined contributions, embedded options, risk sharing, portfolio selection, stochastic programming.

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1 Introduction

Developed and developing countries are currently debating comprehensive approaches for delivering adequate, sustainable and safe retirement incomes to their ageing populations. Defined benefits (DB) pension plans, desirable as they may be for retirees, are not sustainable and all the risks are assumed by the provider, be it a corporate employer or future taxpayers. A consensus has emerged that retirees will “rely more on complementary retirement savings”, Commission (2012), and we are witnessing a shift towards defined contributions (DC). However, these plans pass the risks to the retirees. To make them politically acceptable, encourage participation and increase savings, the retirement income must be safe. Hence, some type of guarantee is needed and the success of DC hinges upon the design of appropriate guarantees. And the difficulty does not stop in designing the guarantee. We then need asset allocation decisions that deliver on the guarantee or appropriate insurance in case the guarantee can not be met. These interrelated decisions need to be “optimised for their safety and performance” in the words of the European Commission report above. Given the complex interactions of financial, economic and demographic risks, a guarantee may fail after all, as much as a “defined benefit” may be modified by government legislation, World Bank (2000). Complementary retirement plans make failures less likely.

In core-DB the provider is faced with a set of rigid promises and assumes all risks. In DC there is no promise made to the beneficiary, who assumes all the risks. This is a binary system. Complementary plans range from DB-lite, i.e., plans with a floor on minimum benefits, to DC-plus, i.e., defined contribution plans with some guarantee on the contributions made during the working life. However, “defined ambition” plans—a term first coined in the Dutch pension reform debate of the 2000’s and currently providing the basis for policy debates in the UK—argues that a tweak to the binary system can not solve the problem and requires better risk sharing to ensure that DC is going to work and be sustainable. A comprehensive approach views pension plans as a hybrid of guarantees and ambitions: nominal annuities are guaranteed, but the degree to which pensions rise in line with prices and wages depends on the performance of the investments of the pension funds. A discussion of the Dutch reforms is given by Bovenberg and Nijman (2009), the papers in NAPF (2012) provides an overview of the issues facing mainly the UK industry in risk sharing, and Smetters (2002) discusses the issues in converting public pensions to DC in the United States.

The contribution of our paper is in modeling DC plans with a variety of guarantee options, in a way that allows ex ante mark-to-market risk premia to facilitate risk sharing. We start with an overview of data that explain the shift from DB to DC, present the world-wide prevalence of DC with guarantees, and discuss the state-of-the-art in pricing and asset management models. This introduction motivates the problem and explains our paper’s novelty.

1.1 The pensions challenge

The Census Bureau reports that before baby boomers started turning 65 in 2010, 11% of the total population in the United States was between the ages of 65 and 84. Thereafter, this age group is projected to reach 18% of the population by 2030, Colby and Ortman (2014). US will experience a 45% increase of ageing population by 2050. Data from the EC, Commission (2012), and the IMF, Carone and Costello (2006), reveal even more

challenging situation in Europe. Older Europeans are a significant and growing part of the EU population (24%) and by 2050 it is projected to grow by 77%. The fastest growing group in the EU are the very elderly (80+) projected to grow by 174%, and the old-age dependency ratio is projected to double to 54%.

At the same period per capita growth rate slides to a projected 1.4%, Carone and Costello (2006), and these are pre-crisis estimates. Pensions represent a large and rising share of public expenditure: more than 10% of GDP on average today, and expected to rising to 12.5% in 2060 in the EU as a whole. Spending on public pensions ranges from 6% of GDP in Ireland to 15% in Italy, so countries are in different situations although they face similar demographic challenges. The projected change between 2004 and 2050 ranges from -5.9% of GDP in Poland to +12.9% of GDP in Cyprus. Only three EU members experience a decrease and nine members expect increases over 5%.

According to the EC white paper for “adequate, safe and sustainable” pensions, Commission (2012), a majority of member states have been reforming pension systems to put them on a more sustainable footing. Shifting from DB to DC is an important dimension of the reforms. An IMF report finds that “on existing [pension] policies, the intertemporal net worth of the EU27 is deeply negative, even in excess of its GDP level, and is projected to worsen further over time. This suggests that Europe’s current policies need to be significantly strengthened to bring future liabilities in line with the EU governments’ capacity to generate assets”, Velculescu (2010).

The challenges are not restricted to the US and EU. Latin American countries were pioneers in pension reforms in the 1990’s; the pricing literature we review below was motivated by DC plans introduced in Uruguay, Chile and Colombia. In India, DB plans were closed by the Government in 2004 and were replaced by a two-tier DC system. The introduction of DC plans in China appears to be modest, but it is a recent ongoing trend. We do not review all developments here but the global trend is evident.

The challenges are addressed with a variety of policy tools: balancing time spent in work and retirement, enhancing productivity, indexing replacement rates, supporting the development of complementary retirement savings to enhance retirement incomes. Shifting away from DB is a way for supporting complementary retirement savings and we focus now on DC plans.

1.2 Type of guarantees

A survey conducted of 1700 organizations in the nine largest EU economies, found 22% of the respondent’s undergoing pension reforms and 42% mentioned sustainability as a reason, Hewitt (2007). In the UK, 88% of DB schemes were open to new members in 2000 but by 2011 this had dropped to 19%, NAPF (2012).

Shifting from DB to DC addresses sustainability issues, but shifts all risks to the beneficiaries. To mitigate risks DC plans typically offer some type of guarantee. In the core DB, the provider commits to a set of rigid promises and takes all the risk. A plain vanilla DC guarantees the nominal value of the contributions.

However, this is not simply a binary system. There is an array of interesting and highly relevant hybrid forms with success stories in Sweden, Denmark or Holland. Hybrid schemes come with a variety of guarantees, such as cash balance pensions where the employer provides a return guarantee on the pension pot but does not guarantee what that will buy in terms of income. From the Hewitt survey 50% of the plans were DB, 32% DC and 18% hybrid. From the NAPFA data only 8% define themselves as hybrid.

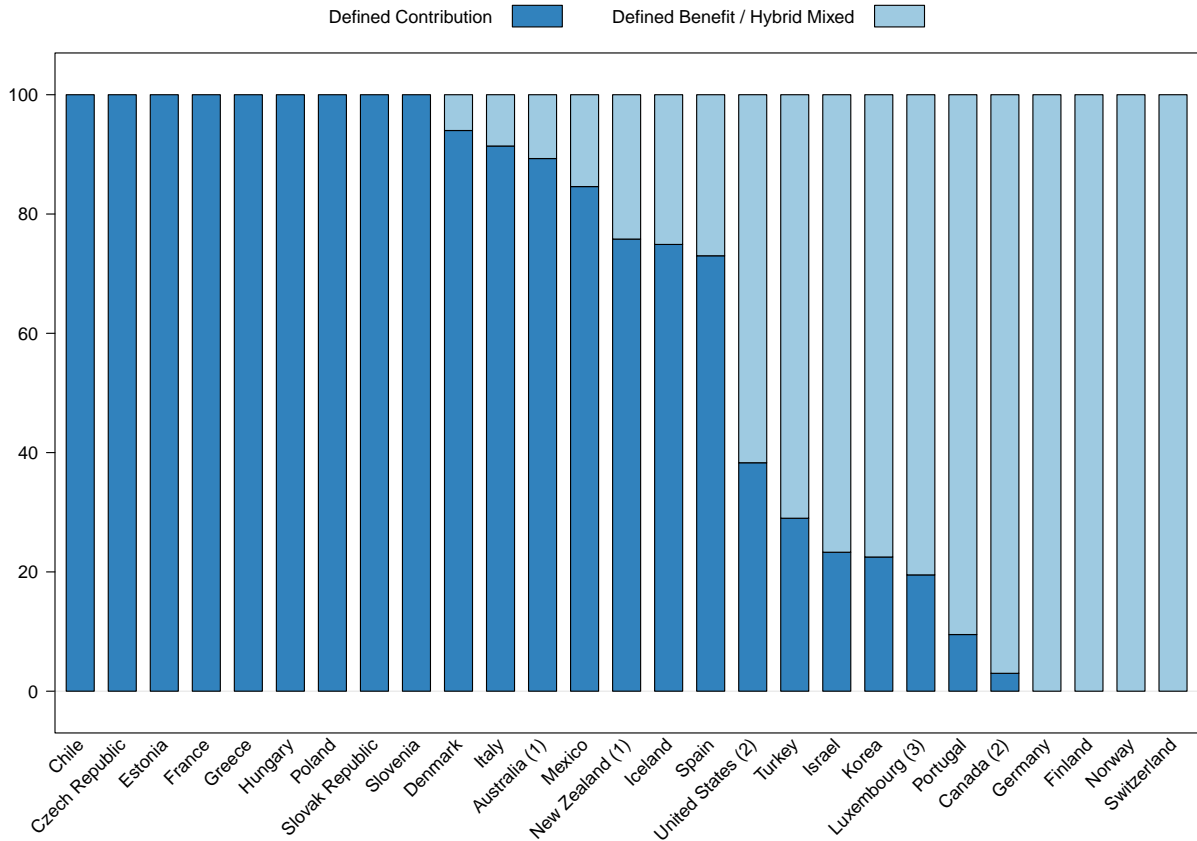


Figure 1: Relative shares of DB, DC and hybrid pension fund assets in selected OECD countries, 2011. Source: OECD Global Pension Statistics, <http://dx.doi.org/10.1787/888932908079>.

In a paper pricing the cost of public pension liabilities in the US, Biggs (2011) uses the database from the National Association of State Retirement Administrators covering 125 mostly state-level programs and finds that around 80% of the employees have a DB pension, 14% DC and 6% have both. Figure 1 illustrates the prevalence of DC and hybrid plans in OECD countries.

Guarantees come in various forms, see, e.g., Antolín (2011). There are significant legislative differences among countries on who backs the guarantee, such as the Government, the provider, a public pension protection fund, a collective DC trust and so on. Leaving aside institutional issues we can classify guarantees based on the level of protection to the beneficiary and the risks to the provider, defining a *risk sharing ladder*:

Rung 1. Money-safe accounts, that guarantee the contributions, either in nominal or real value upon retirement.

Rung 2. Guaranteed return plans, that guarantee a fixed rate of return on contribution, upon retirement.

Rung 3. Guaranteed return to be equal to some industry average upon retirement.

Rung 4. Guaranteed return for each time period until retirement.

Rung 5. Guaranteed income past retirement.

Note an important distinction between the first four and the fifth level of protection. The first four provide the beneficiary with guarantees on level of wealth attained upon retirement while the fifth guarantees retirement income. Of course, wealth accumulation provides the means to buy an income upon retirement, but the connection between the two is not trivial. Plans with the first four levels of protection are focusing on the volatility of assets and returns rather than the risk of not realising inflation-protected incomes. The “Defined Ambitions” debate climbs this risk ladder, offering some protection in the form of guarantees and some in the form of soft guarantees (ambitions).

Deciding how far to climb the risk ladder offers possibilities for risk-sharing, but this requires fair valuation of the risks transferred. For instance, if the employer — or a public protection fund or a collective trust— provides asset volatility insurance for the retirees, the cost of this insurance premium should be determined ex-ante and priced using the markets. Risk transfers should be valued on a mark-to-market basis and whoever underwrites the guarantee —employer, future taxpayers or members of a collective trust— must be compensated (NAPF, 2012, pp. 25–30), and we turn now to the pricing literature.

1.3 Pricing and asset management literature

A minimum guarantee clause can be modeled as a put option written on an underlying *reference portfolio* of assets whose returns determine the return on the contributions. The evaluation of the guarantee option has attracted significant interest from academics, practitioners and policymakers. The seminal papers, developed independently and simultaneously, are Fischer (1999); Pennacchi (1999). Pennacchi used continuous martingale theory to price the guarantees at the second and third rung of our risk ladder. These were the guarantees offered in Uruguay and Chile, respectively. Fischer values Colombia’s guarantees using a discrete martingale model and obtained qualitatively similar results to Pennacchi for the Chilean guarantees. An interesting feature of Fischer’s model is the existence of a ceiling on the guarantee. Pennacchi realised the similarity of pension guarantees with insurance participating products with embedded options priced by Boyle and Hardy (1997); Brennan and Schwartz (1979), see also Embrechts (2000). Advances in this field generated numerous studies whose primary goal is to properly refine the stochastic framework under more general hypotheses, see, e.g., Coleman et al. (2007); Consiglio and De Giovanni (2010), but they apply to the insurance field where the obligation has significant differences from the pension obligation.

There is also extensive literature on pricing the cost of benefit guarantees in DB plans. This is the cost to the public entity (e.g., Public Benefits Guarantee Corporation) that may have to step in to resolve a failing DB plan. This problem is different than the one facing the management of DC plans, as pointed out by Fischer. A discussion of the cost of different types of guarantees, focusing on policy instead of pricing models is Smetters (2002). Options pricing for public pension liabilities is given in Biggs (2011), who used data from State governments in the US to find that public pension shortfalls equal an average of 27% of State GDP. These are compelling arguments for shift towards DC.

In all current literature the underlying reference portfolio is assumed given. However, the construction of a reference portfolio is endogenous to the problem. Recognising this limitation Bacinello (2003) reports sensitivity analyses to different model parameters, that would correspond to different asset portfolios.

An alternative strand of literature focuses on the problem of asset and liability man-

agement (ALM). The handbook Zenios and Ziemba (2007) contains several papers on state-of-the-art models for Dutch (ch. 18) and Swiss (ch. 20) pension funds, the Russell-Yasuda-Kasai model for insurance as adapted for pension funds (ch. 19) and a paper for simultaneously determining asset allocation and contribution rates (ch. 21). These papers follow a parallel stream of ALM models for the insurance industry, such as the seminal model developed for Japanese insurance, the Russell-Yasuda-Kasai model of Cariño et al. (1998); Carinõ and Ziemba (1998). Other examples include the Towers Perrin model of Mulvey and Thorlacius (1998) and the Gjensidige Liv model of Høyland (1998).

In the context of insurance products with guarantees, the problem of structuring the reference portfolio on which the guarantee is written was formulated as a stochastic programme in the *PROMETEIA* model, see ch. 15 in the handbook, and was applied to UK and Italian policies, Consiglio et al. (2006, 2008). This model addressed an important issue since the liabilities from the guarantee—and bonus payments for insurance products—depend on the asset portfolio, but while it extended ALM literature to account for guarantees it did not price the guarantee per se.

In this paper we endogenize the decision about the reference portfolio in a model for pricing the guarantee. We provide a model that extends the line of research started by Pennacchi–Fischer to select a reference portfolio that minimises the cost of the guarantee, whereby the cost is obtained using option pricing theory on a discrete tree.

Finding the minimum cost of an option is, computationally, a very complex problem because the option payoff is a non-smooth function of the endogenous portfolio choice. Moreover, if a pricing formula is not available in closed-form, the large number of scenarios needed to represent the underlying stochastic processes makes the optimization problem intractable due to the curse of dimensionality. The model setup we develop in this paper is innovative in two aspects. First, it integrates portfolio optimization and option pricing in a unified framework that is computationally tractable. Second, it provides a strategic tool to compare alternative guarantee designs to a yardstick represented by reference portfolio with minimal guarantee costs thereby facilitating risk sharing. The model is tractable for large-scale instances.

2 The model setup

We assume that asset returns are stochastic processes in discrete space and time. The set of asset returns is labeled by index set $\mathcal{J} = \{1, 2, \dots, J\}$ and are observed at finite time instances $t \in \mathcal{T}$, where $\mathcal{T} = \{1, 2, \dots, T\}$:

$$R = (R_t^1, \dots, R_t^J)_{t=1}^T. \quad (1)$$

The return process is modeled on the probability space (Ω, \mathcal{F}, P) , where the sample space Ω is assumed to be finite. Such a formulation allows for a market representation through *scenario trees*, Pliska (1997). We denote by \mathcal{N}_t the set of nodes at t , and by $\mathcal{N} \equiv \bigcup_{t=0}^T \mathcal{N}_t$ the collection of all the nodes. Each node $n \in \mathcal{N}_t$ corresponds one-to-one with an atom of the filtration \mathcal{F}_t . (For simplicity, whenever we refer from now on to a node n , it is understood to be a node from the set \mathcal{N}_t for all t , unless specified otherwise.) These are possible future states of the economy at time t . Not all nodes at t can be reached from every node at $t - 1$ and we define *paths* from the *root node* 0 to some final node in the set \mathcal{N}_T to denote the unique way of reaching a particular node. Each path

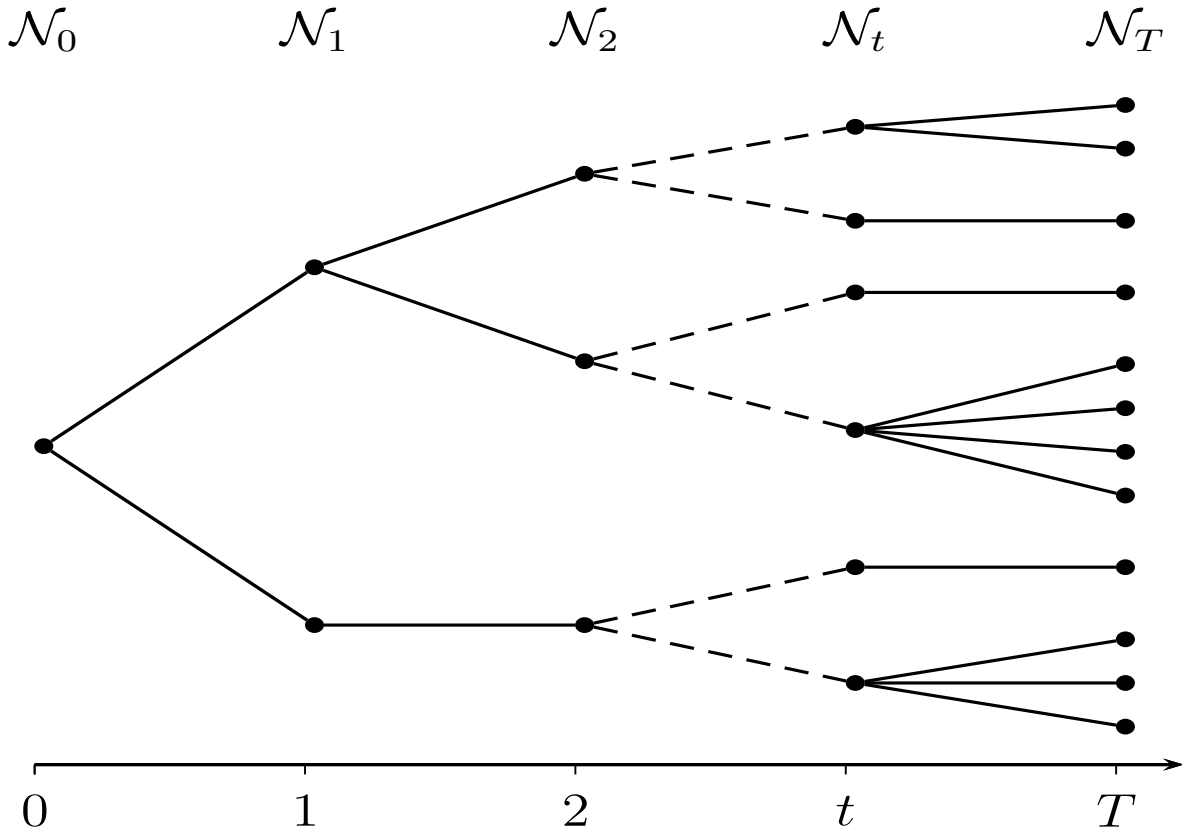


Figure 2: A scenario tree.

is a *scenario*¹. A graphical representation of a tree is given in Figure 2, for an example with 12 scenarios, two possible states at $t = 0$, three possible states at $t = 1$ and six at t . We denote by $\mathcal{P}(n)$ the set of nodes on the unique path from the root node to $n \in \mathcal{N}_t$, by $p(n)$ the unique predecessor node for n , with $p(0)$ being empty, and by $\mathcal{S}(n) \subset \mathcal{N}_t$ the non-empty set of successor nodes. Each successor node is associated with a weight q_n , interpreted as a probability. Given n , all information contained in path $\mathcal{P}(n)$ is known.

2.1 The basic minimum guarantee option

We assume that a DC fund guarantees a payment at maturity T . The price of this guarantee is contingent on the value of a reference fund A_n for each $n \in \mathcal{N}_T$. In the basic model we assume a closed fund with initial total contributions L_0 . Typically, there will be some regulatory equity requirement² E_0 , such as $E_0 = (1 - \alpha)A_0$ and $\alpha < 1$, so that $L_0 = \alpha A_0$. The initial endowment $A_0 = L_0 + E_0$ is invested in a reference portfolio according to the asset allocation proportional variables x_j , $\sum_{j \in \mathcal{J}} x_j = 1$.

Given the family of stochastic processes $\{R_t\}_{t \in \mathcal{T}}$, defined as a J -dimensional vector

¹Monte Carlo simulations can also be represented through scenario trees. In such a case, the trees are *scenario fans* and each non-final node has only one successor.

²EU Directive 2009/138/EC known as ‘‘Solvency II’’, becoming effective in January 2016, sets forth rigorous regulatory requirements for insurance and re-insurance, available at <http://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:02009L0138-20140523&from=EN>. No similar directive is yet in place for pension funds but for generality of the model we introduce the initial equity parameter.

of returns, $R_n \equiv (R_n^1, \dots, R_n^J)$, the stochastic process of the asset value $\{A_n\}$ is driven by the stochastic process of the portfolio return $\{R_n^A\}$ for each $n \in \mathcal{N} \setminus \{0\}$ as follows:

$$A_n = A_{p(n)} e^{R_n^A}, \quad (2)$$

$$R_n^A = \sum_{j \in \mathcal{J}} x_j R_n^j, \quad (3)$$

where $A_{p(n)}$ is the asset value at the predecessor node. The main difference with known guarantee models reviewed earlier is that the reference portfolio performance depends on the asset allocation decisions x_j .

The liability process $\{L_n\}$ also grows at a stochastic rate that, at each node $n \in \mathcal{N} \setminus \{0\}$, is guaranteed not to be less than a *minimum guarantee* rate g . That is,

$$R_n^L = \max(\delta R_n^A - g, 0) + g, \quad (4)$$

where δ is the *participation rate* and denotes the fraction of risky return which is passed to the beneficiary. Typically δ would be 1 net management fees, but it can also be set to lower values to compensate the fund for the offered guarantee by keeping a fraction of the portfolio upside to compensate for providing a guarantee in a downturn.

The stochastic process of the liability is given, for all $n \in \mathcal{N} \setminus \{0\}$, by:

$$L_n = L_{p(n)} \exp [g + \max(\delta R_n^A - g, 0)], \quad (5)$$

where $L_{p(n)}$ is the liability value at the predecessor node. For simplicity we ignore the effect of mortality on the liability process. This can be easily accommodated by introducing an intensity based model, where the survival probability at node n is a deterministic function of time, or by fitting a stochastic mortality process to the scenario tree.

3 The optimization model

We now formulate the mathematical program to determine the optimal composition of the reference portfolio, and show how to reduce its computational complexity.

3.1 The objective function

Denote by $\Phi(A_n, L_n)$ the payoff function measuring the performance of the portfolio strategy at each final node n . A_n and L_n are implicit functions of the asset allocation x_j , and according to options theory the price of this contingent payoff is given by

$$\Gamma = e^{-rT} \sum_{n \in \mathcal{N}_T} q_n \Phi(A_n, L_n), \quad (6)$$

where the expectation is discounted at the risk free rate r , and is taken under the *risk-neutral measure*. In our discrete probabilistic setting these are the weights q_n , $n \in \mathcal{N}_T$.

To explicitly write the objective function, we must specify the payoff function $\Phi(A_n, L_n)$. A natural choice is

$$\Phi(A_n, L_n) = \max(L_n - A_n, 0), \quad (7)$$

that implicitly assumes that shareholders cover possible shortfalls from the guarantee at each state. In this respect, a rationale strategy for the fund management is to minimizing the expected value of these losses. Substituting in (6) we obtain

$$\Gamma = e^{-rT} \sum_{n \in \mathcal{N}_T} q_n \max(L_n - A_n, 0), \quad (8)$$

and this is the cost of a put option written on the value of the assets A_n with a stochastic strike price L_n , i.e., it is the cost of the guarantee.

This cost can be considered as the fair risk premium charged by a pensions guarantee agency to the fund, to set minimum equity requirements by a regulator, to determine charges to the internal sub-division operating the specific DC fund, or to determine prices for risk-sharing between the beneficiary and the fund. The guarantee option contributes to the total risk of the company and minimizing its value is a consistent with enterprise-wide risk management.

3.2 The bilinear constraints

The variables affecting the cost of the guarantee (8) are the values of the asset and liability accounts at T . We denote by w_n and z_n , respectively, the final cumulative returns of the asset and liability accounts A_n and L_n . For all $n \in \mathcal{N}_T$, we have that:

$$w_n = \sum_{i \in \mathcal{P}(n)} R_i^A, \quad (9)$$

$$z_n = \sum_{i \in \mathcal{P}(n)} g + \max(\delta R_i^A - g, 0), \quad (10)$$

where, with a slight abuse of notation, $\mathcal{P}(n)$ does not include the root node.

The max operator in eqn. (10) implies that the problem is a discontinuous nonlinear programming model (DNLP), thus making the optimization model intractable for large-scale applications. To solve the DNLP we reformulate it as an equivalent smooth nonlinear program. We can write the first argument of the max operator as the difference of two positive variables, with the condition that only one of them is non-zero. Thus, for all $n \in \mathcal{N} \setminus \{0\}$, we introduce the following set of equations:

$$\delta R_n^A - g = \varepsilon_n^+ - \varepsilon_n^-, \quad (11)$$

$$\varepsilon_n^+ \varepsilon_n^- = 0, \quad (12)$$

$$\varepsilon_n^+, \varepsilon_n^- \geq 0. \quad (13)$$

The bilinear constraints (12) still add complexity that is difficult to deal with when we wish to solve models with a large number of nodes on the tree. However, we prove in the next section that it is not necessary to explicitly add constraints (12).

Given eqn. (11), we rewrite the definitional eqn. (10) as follows:

$$z_n = gT + \sum_{i \in \mathcal{P}(n)} \varepsilon_i^+. \quad (14)$$

With this notation, the final value of assets and liabilities is given by

$$A_n = A_0 e^{w_n}, \quad (15)$$

$$L_n = \alpha A_0 e^{z_n}, \quad (16)$$

for each node $n \in \mathcal{N}_T$, and recall that $L_0 = \alpha A_0$.

We also need to reduce the complexity introduced by the max operator in the payoff function (7). To do so, we must handle the exponentiation terms of w_n and z_n . Let us consider the generic n -th term of the summation in eqn. (8). Substituting the expressions (15)–(16) in place of A_n and L_n , we obtain:

$$\begin{aligned} \max(L_n - A_n, 0) &= A_n \left[\max\left(\frac{L_n}{A_n}, 1\right) - 1 \right] \\ &= A_0 e^{w_n} \left[\max(\alpha e^{z_n - w_n}, 1) - 1 \right]. \end{aligned} \quad (17)$$

If $a, b > 0$, then $\max(a, b) = e^{\max(\ln a, \ln b)}$ and we write the last term in (17) as

$$A_0 e^{w_n} \left[e^{\max(\ln \alpha + z_n - w_n, 0)} - 1 \right]. \quad (18)$$

We can now redefine the max operator as a set of constraints by writing its first argument as the difference of two positive variables

$$\ln \alpha + z_n - w_n = H_n^+ - H_n^-, \quad (19)$$

$$H_n^+ H_n^- = 0, \quad (20)$$

$$H_n^+, H_n^- \geq 0, \quad (21)$$

and expression (18) becomes

$$A_0 e^{w_n} \left(e^{H_n^+} - 1 \right). \quad (22)$$

The bilinear equations (20) also lead to intractable optimization problems, but as we will prove later these constraints again need not be added explicitly.

Given (22), the cost of the guarantee (8) becomes:

$$\Gamma(x_1, x_2, \dots, x_J) = e^{-rT} A_0 \sum_{n \in \mathcal{N}_T} q_n e^{w_n} \left(e^{H_n^+} - 1 \right). \quad (23)$$

We now show that (23) is a convex function of the portfolio choices x_j .

Proposition 1. *The function Γ is a convex function of the portfolio choices x_j , $j \in \mathcal{J}$.*

Proof. By the definitions of H_n^+ and w_n , it is easy to show that these quantities are affine functions of x_j , $j \in \mathcal{J}$, then e^{w_n} and $(e^{H_n^+} - 1)$ are convex functions of x_j , $j \in \mathcal{J}$. Since e^{w_n} and $(e^{H_n^+} - 1)$ are increasing and non-negative functions of w_n and H_n , then their product is convex. Finally, given that the coefficients q_n , $n \in \mathcal{N}_T$, are probabilities, then Γ is a linear combination of convex functions and it is also convex. \square

3.3 Convex model for minimizing the cost of guarantee option

If bilinear equations (12) and (20) are omitted, the remaining constraints are linear. Moreover, given the convexity of the objective function, the minimization of the guarantee option is a convex programming problem. In convex optimization if a local minimum exists then it is a global minimum, and effective computational methods are available, even for large scale instances, using off-of-the-shelf optimization packages. Hence, the

portfolio which minimizes the cost of the guarantee option is obtained as the solution to the following convex non-linear program:

Problem 1. Convex optimization of the guarantee option

$$\text{Minimize}_{x_1, \dots, x_J} e^{-rT} A_0 \sum_{n \in \mathcal{N}_T} q_n e^{w_n} \left(e^{H_n^+} - 1 \right) \quad (24)$$

s.t.

$$\ln \alpha + z_n - w_n = H_n^+ - H_n^-, \quad n \in \mathcal{N}_T, \quad (25)$$

$$\delta R_n^A - g = \varepsilon_n^+ - \varepsilon_n^-, \quad n \in \mathcal{N} \setminus \{0\}, \quad (26)$$

$$z_n = gT + \sum_{i \in \mathcal{P}(n)} \varepsilon_i^+, \quad n \in \mathcal{N} \setminus \{0\}, \quad (27)$$

$$w_n = \sum_{i \in \mathcal{P}(n)} R_i^A, \quad n \in \mathcal{N} \setminus \{0\}, \quad (28)$$

$$R_n^A = \sum_{j \in \mathcal{J}} x_j R_n^j, \quad n \in \mathcal{N} \setminus \{0\}, \quad (29)$$

$$\sum_{j \in \mathcal{J}} x_j = 1, \quad (30)$$

$$H_n^+, H_n^- \geq 0, \quad n \in \mathcal{N}_T, \quad (31)$$

$$\varepsilon_n^+, \varepsilon_n^- \geq 0, \quad n \in \mathcal{N} \setminus \{0\}, \quad (32)$$

$$x_j \geq 0, \quad j \in \mathcal{J}. \quad (33)$$

To establish the validity of this model we need to prove that the minimum value of the objective function of Problem 1, coincides with that of the non-convex problem obtained by adding the nonlinear equations (12) and (20) to the constraints set (25)–(33).

Lemma 1. *Let us assume that $x_1^*, x_2^*, \dots, x_J^*$ is an optimal portfolio choice for Problem 1. Then,*

$$H_n^+ H_n^- = 0,$$

for all $n \in \mathcal{N}_T$.

Proof. We will prove the lemma by negating the thesis and showing that this contradicts the hypothesis that $x_1^*, x_2^*, \dots, x_J^*$ is a minimum of Problem 1.

Let us assume that there exists a $k \in \mathcal{N}_T$ such that $H_k^+ H_k^- > 0$. We define $\lambda = \min(H_k^+, H_k^-)$. Since $\lambda > 0$, subtracting λ from both H_k^+ and H_k^- will keep the non-negativity of both variables, modify the objective function, and leave unaffected the rest of variables and constraints.

Let us denote by $\Gamma(\lambda)$ the optimal level of the objective function after we have subtracted λ from H_k^+ and H_k^- . We have:

$$\Gamma(\lambda) = e^{-rT} A_0 \left[\sum_{\substack{n \in \mathcal{N}_T \\ n \neq k}} q_n e^{w_n} \left(e^{H_n^+} - 1 \right) + q_k e^{w_k} \left(e^{H_k^+ - \lambda} - 1 \right) \right].$$

By simple algebra, it is possible to show that the change in the optimal objective function due to λ , $\Delta\Gamma = \Gamma(\lambda) - \Gamma$, is given by:

$$\Delta\Gamma = e^{-rT} A_0 q_k e^{w_k} e^{H_k^+} \left(e^{-\lambda} - 1 \right).$$

We observe that $\Delta\Gamma < 0$ since $(e^{-\lambda} - 1) < 0$ and the other terms are all positive. Therefore, the objective function can be further reduced, but this contradicts the main hypothesis that $x_1^*, x_2^*, \dots, x_J^*$ is a minimum for Problem 1. Hence, the assumption that there exists a k such that $H_k^+ H_k^- > 0$ must be false. \square

Remark 1. Lemma 1 only ensures that if an optimal solution of Problem 1 exists, then (20) holds for all $n \in \mathcal{N}_T$. To claim that such a minimum coincides with that of the non-convex problem, we have to prove that at the minimum of Problem 1 also conditions (12) hold. If the latter result is true, since the objective function is convex, and therefore the minimum level of the objective function is unique, we can conclude that the minimum of the convex and the non-convex problem are the same. In general, however, this is not true for the optimal portfolio $x_1^*, x_2^*, \dots, x_J^*$. Only strict convexity of the objective function would imply the uniqueness of the optimal portfolio choices.

Lemma 2. *Let us assume that $x_1^*, x_2^*, \dots, x_J^*$ is an optimal portfolio choice for Problem 1. Then, it exists a non empty subset of nodes $\mathcal{B} \subset \mathcal{N}$ such that $\forall n \in \mathcal{B}$ we have*

$$\varepsilon_n^+ \varepsilon_n^- = 0.$$

Proof. Let us assume that for all $n \in \mathcal{N}$ we have $\varepsilon_n^+ \varepsilon_n^- > 0$. We denote by $\xi = \min(\{\varepsilon_n^+\}_{n \in \mathcal{N}} \cup \{\varepsilon_n^-\}_{n \in \mathcal{N}})$ and subtract ξ from ε_n^+ and ε_n^- , for all $n \in \mathcal{N}$. Such a subtraction will affect the objective function value and the constraints (25). We define by $\lambda = \xi T$ the total change in the left-hand-side of the constraints set (25), where T is the number of time steps. In particular, we have:

$$\ln \alpha + z_n - \lambda - w_n = H_n^+(\lambda) - H_n^-(\lambda), \text{ for all } n \in \mathcal{N}_T.$$

The effect of λ is counterbalanced by a change in the variables H_n^+ and H_n^- . By Lemma 1, we have that $H_n^+ H_n^- = 0$ at the optimum. Let us assume that the change due to λ preserves such a property, and therefore, $H_n^+(\lambda) H_n^-(\lambda) = 0$, for all $n \in \mathcal{N}_T$. The final nodes are then partitioned as follows:

- if $H_n^+ > \lambda$, then $H_n^+(\lambda) = H_n^+ - \lambda$ and $H_n^-(\lambda) = 0$, for all $n \in \mathcal{N}_T^a \subset \mathcal{N}_T$;
- if $H_n^+ > 0$ and $H_n^+ \leq \lambda$, then $H_n^+(\lambda) = 0$ and $H_n^-(\lambda) \geq 0$, for all $n \in \mathcal{N}_T^b \subset \mathcal{N}_T$;
- if $H_n^- > 0$, then $H_n^-(\lambda) = H_n^- + \lambda$ and $H_n^+(\lambda) = 0$, for all $n \in \mathcal{N}_T^c \subset \mathcal{N}_T$.

The objective function $\Gamma(\lambda)$ is then partitioned accordingly³:

$$\Gamma(\lambda) = \sum_{n \in \mathcal{N}_T^a} q_n e^{w_n} \left(e^{H_n^+ - \lambda} - 1 \right) + \sum_{n \in \mathcal{N}_T^b} q_n e^{w_n} (e^0 - 1) + \sum_{n \in \mathcal{N}_T^c} q_n e^{w_n} (e^0 - 1),$$

and the change in the optimal objective function, $\Delta\Gamma = \Gamma(\lambda) - \Gamma$, is given by:

$$\Delta\Gamma = \sum_{n \in \mathcal{N}_T^a} q_n e^{w_n} e^{H_n^+} (e^{-\lambda} - 1) - \sum_{n \in \mathcal{N}_T^b} q_n e^{w_n} (e^{H_n^+} - 1).$$

Observe that $\Delta\Gamma < 0$ since it is made up by two negative terms (the summation over the set of nodes \mathcal{N}_T^c is not displayed as it has a null impact in the total change of

³We drop the constant term $e^{-rT} A_0$ for better readability.

the objective function). This contradicts the hypothesis that $x_1^*, x_2^*, \dots, x_J^*$ is an optimal portfolio choice for Problem 1, and then, for each $k \in \mathcal{B}$ it must be that $\varepsilon_k^+ \varepsilon_k^- = 0$, where \mathcal{B} is the union of the nodes belonging to $\mathcal{P}(n)$, with $n \in \mathcal{N}_T^a \cup \mathcal{N}_T^b$. In symbol:

$$\mathcal{B} = \bigcup_{n \in \mathcal{N}_T^a \cup \mathcal{N}_T^b} \mathcal{P}(n). \quad (34)$$

□

Remark 2. A further result following from Lemma 2 is that, at the optimum, the set of nodes $k \in \mathcal{B}$ correctly defines the expression $\max(\delta R_n^A - g, 0)$ since, as proved, $\varepsilon_n^+ \varepsilon_n^- = 0$. Moreover, they belong to paths that lead to final nodes characterized by an $H_n^+ \geq 0$. This implies that the optimal objective function value Γ^* is correctly computed.

On the contrary, the rest of the nodes $k \in \mathcal{N} \setminus \mathcal{B}$, where the max operator is not properly defined since it might occur that $\varepsilon_n^+ \varepsilon_n^- > 0$, belongs to paths leading to final nodes for which H_n^- is surely greater than zero (or at most $H_n^- = H_n^+ = 0$), thus making null their contribution in the computation of Γ^* .

Corollary 1. *Let $x_1^*, x_2^*, \dots, x_J^*$ be an optimal portfolio choice for Problem 1, if $\varepsilon_k^+ \varepsilon_k^- > 0$, for any $k \in \mathcal{N}$, then it exists $n \in \mathcal{N}_T$ such that $k \in \mathcal{P}(n)$ and*

$$H_n^- > 0 \text{ or } H_n^- = H_n^+ = 0.$$

We now assemble the above results to prove that solving Problem 1 is equivalent to solving the same problem with the additional non-convex constraints (12) and (20).

Theorem 1. *Let $x_1^*, x_2^*, \dots, x_J^*$ be an optimal portfolio choice for Problem 1, with optimal objective value Γ^* . Let $x_1^{**}, x_2^{**}, \dots, x_J^{**}$ be an optimal portfolio choice of Problem 1 with the non-convex equations (12) and (20) included, with optimal objective value Γ^{**} . Then*

$$\Gamma^* = \Gamma^{**}.$$

Proof. Lemma 1 ensures that the conditions $H_n^+ H_n^- = 0$ holds for all $n \in \mathcal{N}_T$. Lemma 2 implies that the conditions $\varepsilon_k^+ \varepsilon_k^- = 0$ only hold partially (for all $k \in \mathcal{B}$). However, by Corollary 1 we can claim that if $\varepsilon_k^+ \varepsilon_k^- > 0$ then $k \in \mathcal{P}_n$, i.e. the path leading to $H_n^- > 0$. So, even if $\max(\delta R_k^A - g, 0)$ is not properly defined, its contribution to the value of Γ^* is nil. Finally, the convexity of the objective function implies that its optimal value is unique, hence, $\Gamma^* = \Gamma^{**}$. □

3.4 Extensions

How does the model fit the risk ladder described in the introduction? The current model is for Rung 4, but it can be modified to represent any one of the other steps. Some modifications are straightforward while others are more elaborate and are only sketched here as directions for further research.

Rung 1. For money-safe accounts we eliminate eqn. (4) from the model. The stochastic process of the liability is simplified to

$$L_n = L_{p(n)} \exp [R_n^A], \quad (35)$$

and constraints $L_n \geq L_0$ are added for each terminal node $n \in \mathcal{N}_T$. The resulting model is a simplified version of the model developed above.

Rung 2. To guarantee total return g_L upon retirement, eliminate eqn. (4), use the simplified liability process (35), and add constraints for each terminal node $n \in \mathcal{N}_T$:

$$L_n \geq g_L L_0. \quad (36)$$

The resulting model is also a simplified version of the model developed above.

Rung 3. To deliver the industry average upon retirement, replace g by g_{L_n} in eqn. (36), where g_{L_n} is the state-dependent industry average. To implement this model we need to estimate the industry average portfolio return. Given scenarios of returns on the tree, this information is computed from the composition of industry portfolios. The resulting model is structurally identical to the model developed above but it has additional data requirements.

Rung 4. This is the type of guarantee already modeled. One significant extension is to open funds, whereby in addition to the original endowment L_0 additional contributions I_n are made to the fund at $t = 2, 3, \dots, T$. This is trivially modeled on the liability side by rewriting the stochastic process as:

$$L_n = L_{p(n)} \exp [g + \max (\delta R_n^A - g, 0)] + I_n. \quad (37)$$

The contributions must also be accounted for on the asset side. A simple approach is to assume that incremental contributions are invested proportionately with the original portfolio. A more complex, and more realistic, approach is to rebalance the portfolio as new contributions are made. The model setup on a multi-period tree permits the extension. Indeed, an advantage of the model setup is that it extends to multi-period optimization. This is an active research area, Zenios (2007), and has been used successfully in the ALM literature cited earlier, Carinõ and Ziemba (1998); Høyland (1998); Mulvey and Thorlacius (1998). However, the linearizations developed in this paper need to be re-worked.

Rung 5. To model guarantee income past retirement the model needs a past-retirement horizon $T' > T$. The asset at each state n of the retirement horizon T can then be used to finance income for all nodes emanating from n until T' , and lower bounds imposed on what this income would be. However, it is not clear what additional non-linearities may be introduced with this extension and how to linearise them. Also, multi-factor trees are needed to calibrate risk-neutral probabilities of financial variables together with objective probabilities of economic variables to model inflation-adjusted income, but such trees are available, see, e.g., Consiglio et al. (2015).

4 Implementation and results

We run our experiments for $T = 30$ years and $J = 12$ financial asset indices. Without loss of generality—and appropriately in the current economic environment—we set the risk free rate $r = 0$. The indices represent the broad asset classes of sovereign bonds, corporate bonds and stocks; see Appendix A. We simulate the risk-neutral process of asset returns using a standard Montecarlo approach. The variance-covariance matrix is estimated from the monthly historical series of the indices. The yearly equivalent volatilities are obtained using the square root rule. All data are summarised in Appendix A.

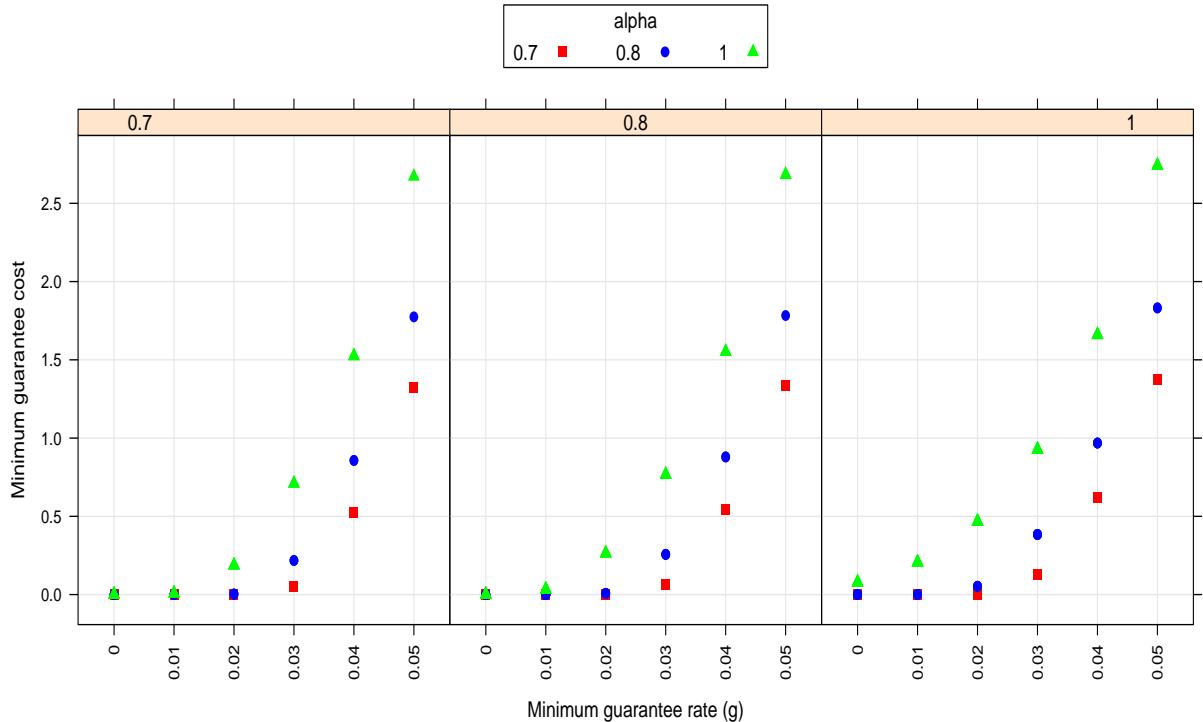


Figure 3: Effect of guaranteed rate g on the cost of the guarantee Γ^* for different values of α . The values of δ are indicated in the strip above each panel.

The model was implemented on a simulated *fan* of 1,000 risk-neutral paths. The size of the scenario set is chosen to limit computational times for the extensive experiments we run. Problems with more scenarios are solvable with modest computer resources, and variance reduction techniques can reduce sampling errors with fewer scenarios. While we use a simple scenario generation method, we point out that alternative discrete representations, e.g., Consiglio et al. (2015); Geyer et al. (2010), can be readily implemented since the mathematical program is formulated on a general tree structure.

The model is tested for different values of the parameters. We let α and δ vary between 0.7 and 1.0, and g to vary between 0% and 5%.

4.1 The effect of policy parameters on the cost of the guarantee

The relationship between the cost of the guarantee Γ^* and the guarantee rate g is shown in Figure 3. Within each panel, we display the effect of the parameter α on the cost, where higher values correspond to less equity and for $\alpha = 1$ there is no equity. Each panel corresponds to different participation rate δ . The lower the δ the higher is the proportion of portfolio upside kept by fund management and this reduces the cost of the guarantee, especially for higher guarantees and lower equity. Figure 4 shows the changes in the cost of the guarantee for varying participation rates, for $g = 3\%$ and equity 0 and 0.3.

We now focus on the effect of α . This parameter controls risk sharing, as it determines the amount of equity, which can be viewed as regulatory requirement for solvency. For fixed $\delta = 0.9$ we show in Figure 5 the cost of different guarantees as α varies from 0.7 to 1.0. The cost of the guarantee increases with the guarantee, but this increase is lower for lower α , i.e., for more equity. This result is intuitive. By reducing α we increase the

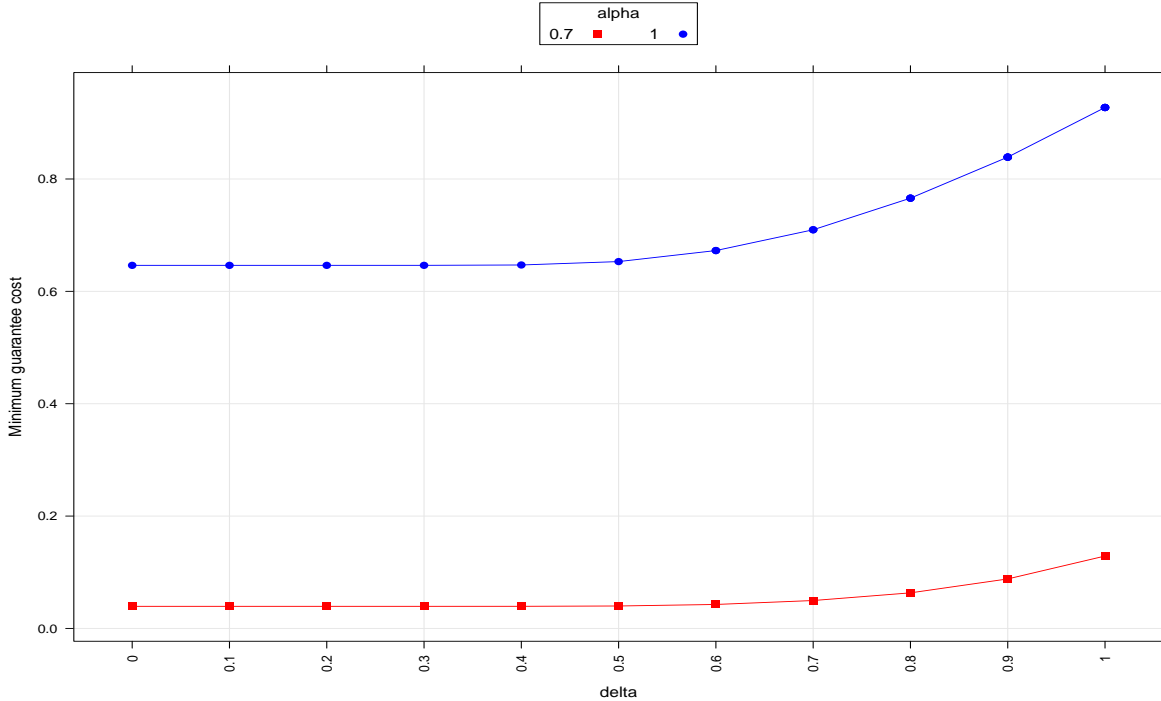


Figure 4: Effect of participation rate δ on the cost of a 3% guarantee for different α .

fraction of equity which composes the initial endowment A_0 . Moreover, from eqn. (16), we observe that the final liability is due only on a fraction α of the initial value of the reference fund, A_0 . Hence, as α decreases an increasing part of the guarantee cost, given by $(L_n - A_n)^+$, is borne by the equity-holders.

The curves in Figure 5 provides a tool to evaluate tradeoff between the option cost and the minimum guarantee. For a fixed value of α , the points on each curve determine a Pareto frontier. For any guarantee rate g the option cost is at its minimum and, given the convexity of the objective function (see Proposition 1), this value is unique. These curves can be used as a yardstick to assess how effective is a given asset portfolio vis-a-vis those on the efficient frontier. Also regulators can assess the risk of each fund. For instance, Solvency II regulations require embedded options to be marked-to-market, and, given the universe of the assets that characterizes the fund's reference portfolio, a shift away from the optimal portfolio is a warning for future failed commitments that should be monitored by regulators. This point is further elaborated in the next subsection.

4.2 Portfolio composition and moral hazard

The portfolio compositions for α set to 0.7 and 0.9, and for changing levels of guaranteed return, are shown in Figure 6. In all instances, the largest proportion is allocated in asset BONDS_1_3 of sovereign bonds with maturities less than 3 years; this is the asset with the lowest volatility, see Table 1.

This allocation is the consequence of the minimum guarantee mechanism. Whenever there is a downside deviation from the guarantee, the liability will increase by a factor that is a function of g and of the prior, compounded, performances. At the same time, the asset account will perform worse than the guarantee, and its value could turn out to

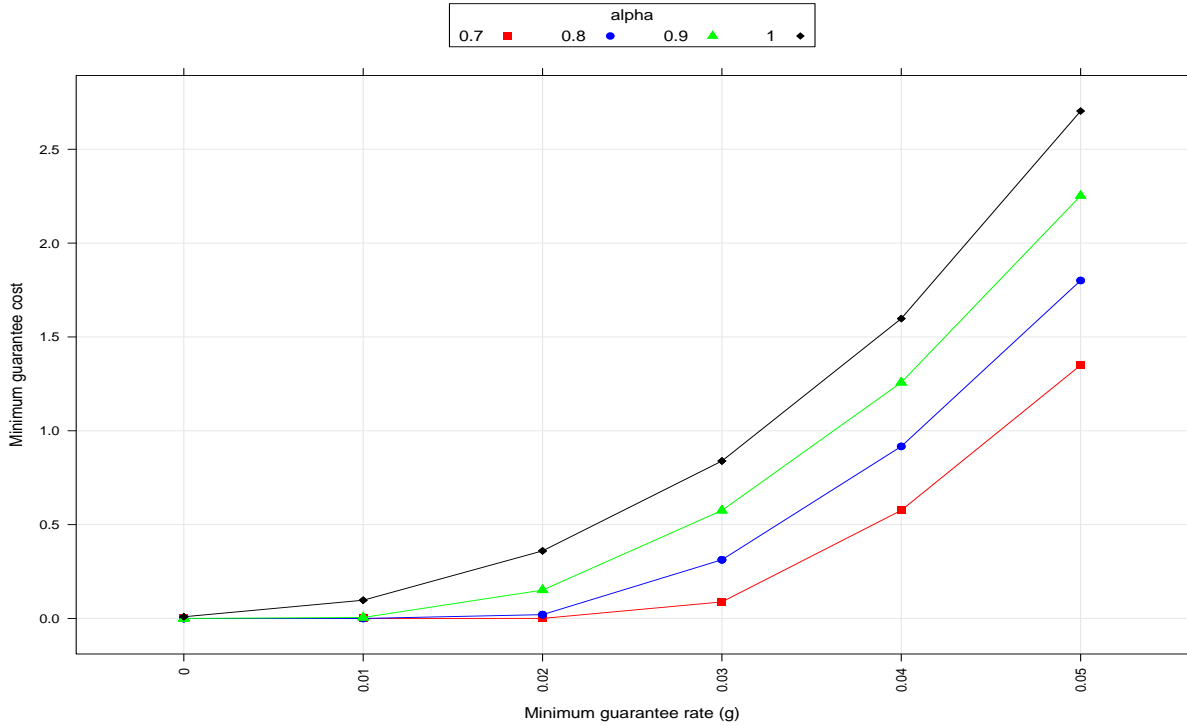


Figure 5: Effect of guaranteed rate g on the cost of the guarantee Γ^* for different values of α . Cost increases with the guarantee, but increase is lower for lower α , i.e., more equity.

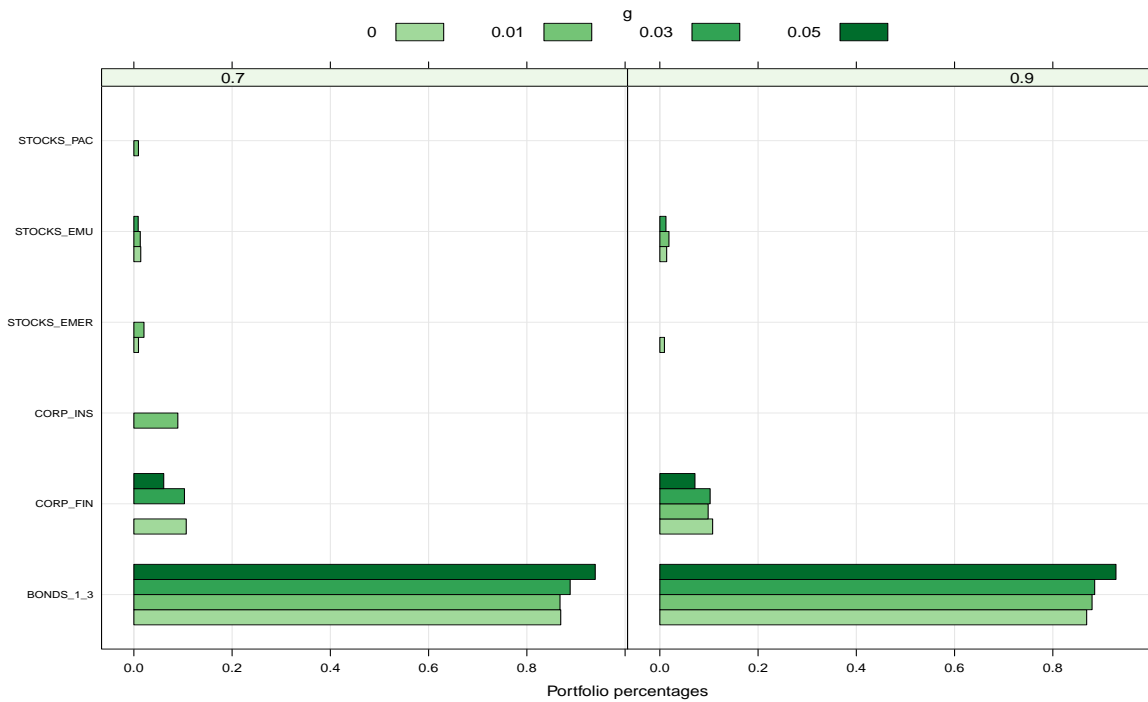


Figure 6: Portfolios are mostly composed of low volatility securities. Participation rate is set $\delta = 0.9$ and α is set to 0.7 and 0.9 as shown in the strip above each panel.

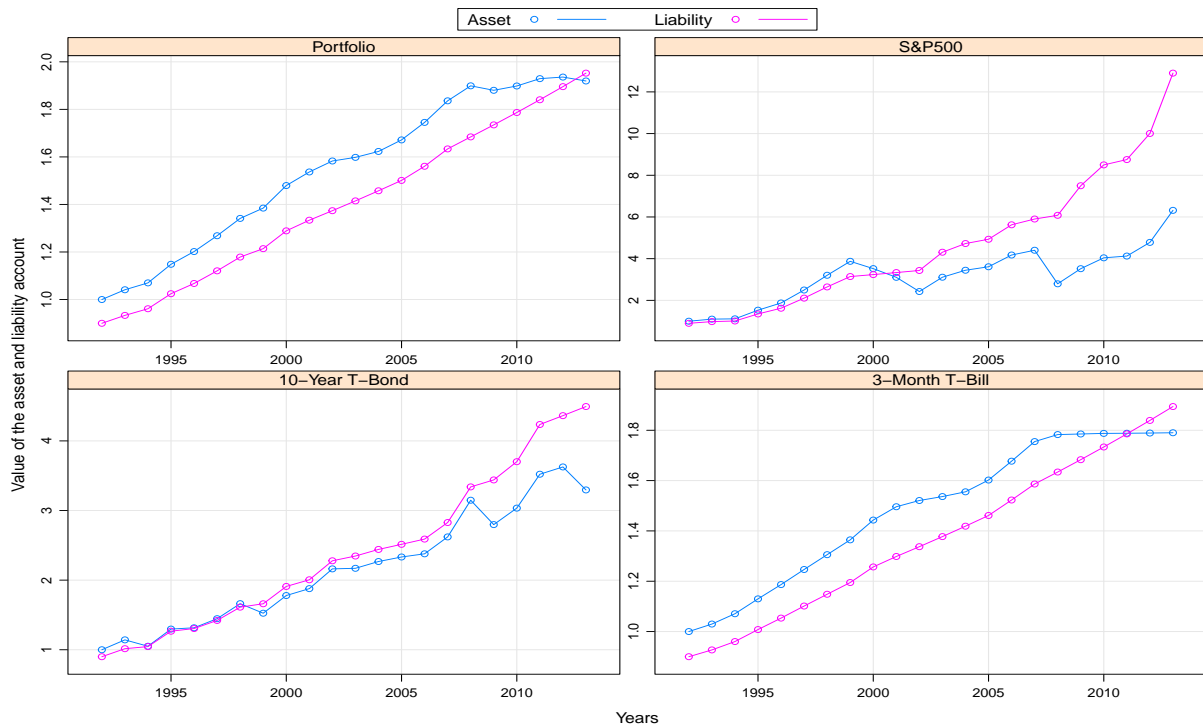


Figure 7: An illustrative example of the dynamics of assets and liabilities of a DC fund with 3% guaranteed return. Results based on historical data of three asset classes and a portfolio with weights $w_1 = 0$, $w_2 = 0.9$ and $w_3 = 0.1$. The less volatile assets and the portfolio better meet the guarantee, consistently with the findings of our model.

be less than the value of the liability implying insolvency. This is illustrated in Figure 7, where four panels show the dynamics of assets and liabilities for $g = 3\%$ and $\alpha = \delta = 0.9$. We consider three asset classes, respectively, the S&P500, the 3-month T-Bill and the 10-year T-Bond, and consider three reference portfolios corresponding to each asset class, plus a portfolio with weights 0, 0.9 and 0.1, respectively. The yearly returns of the four portfolios are shown in Figure 8⁴. Although S&P500 outperforms the minimum guarantee rate, its high volatility leads to a situation in which the final asset value is well below the guaranteed liability. More suitable reference portfolios are obtained by allocating the initial capital over asset classes with low volatilities, or their combination, as shown in Figure 7. This is confirmed by the portfolio results in Figure 6.

The average asset allocations from our model are 90% government bonds, 17% corporates and 3% stocks. By comparison, the average State pension portfolios reported in Biggs (2011) has 58% stocks, 26% bonds, 5% real estate, 2% cash and 9% other. There is significantly more exposure to risky assets in the funds studied by Biggs, compared to the optimal reference portfolio. We use the model to benchmark State pension fund practices. We assume a portfolio of our universe of assets of the same broad asset allocation as the State funds: 60% stocks, 35% bonds and 5% cash, and use the model to price the cost of the guarantee. The results are shown in Figure 9.

Comparing the frontiers of the State sponsored portfolios, with those of the optimised

⁴Data available from <http://www.stern.nyu.edu/~adamodar/pc/datasets/histretSP.xls>. The example is illustrative of the mechanism governing the DC with guaranteed returns. We do not draw any conclusion about the value of the guarantee option that must be addressed under a risk neutral measure.

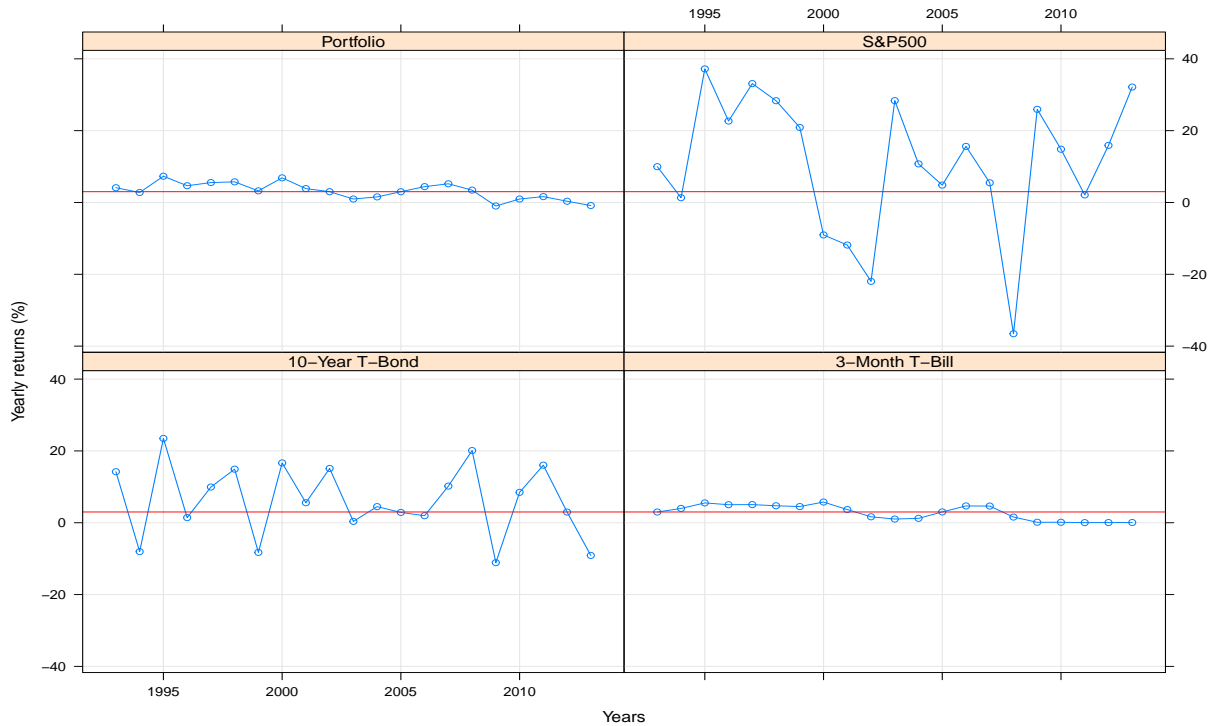


Figure 8: Historical returns of the three assets and the portfolio assumed to be the reference fund of DC fund. The horizontal line denotes the 3% guarantee.

reference fund we observe significantly higher costs for the guarantee of the former. It is not surprising that Biggs finds funding ratios of 45% and concludes that the typical State has significant unfunded public pension liabilities. The differences of cost between the actual and the optimised reference portfolios is a signal of moral hazard. The State guarantees the fund and will have to cover any shortfalls, while the funds take risky positions and pass the upside to the beneficiaries. This could also be a strategy of “gambling for redemption”: underfunded funds take a big gamble hoping for a big upswing.

4.3 Policy implications for risk sharing

The cost of the guarantee can be used to set risk sharing premier. From the example of Figure 5 we obtain for $g=3\%$ and $\alpha = 1$ (zero equity) a cost of 0.84. There are three possible ways to share this cost, depending on the arrangements for risk sharing:

1. The pension fund bears the risk of the guarantee. Under this arrangement, the fund will charge the beneficiary —or the employer, or the government— the cost of the option. Hence, the beneficiary will pay 1.84 and will get a return only on 1 euro, since the 0.84 is the cost of the option. Probably this is not viable, because 3% guarantee is impossible, but the charge is fair. (Of course, since they bear the risk, they have to hedge the option, otherwise they will loose money, or default, even if they get the option premium).
2. There is risk sharing between the beneficiary and a third party —such as the employer or the government. Let us assume that the sharing consist in paying 0.3 of equity ($\alpha=0.7$), and investing it in the reference portfolio. We note from

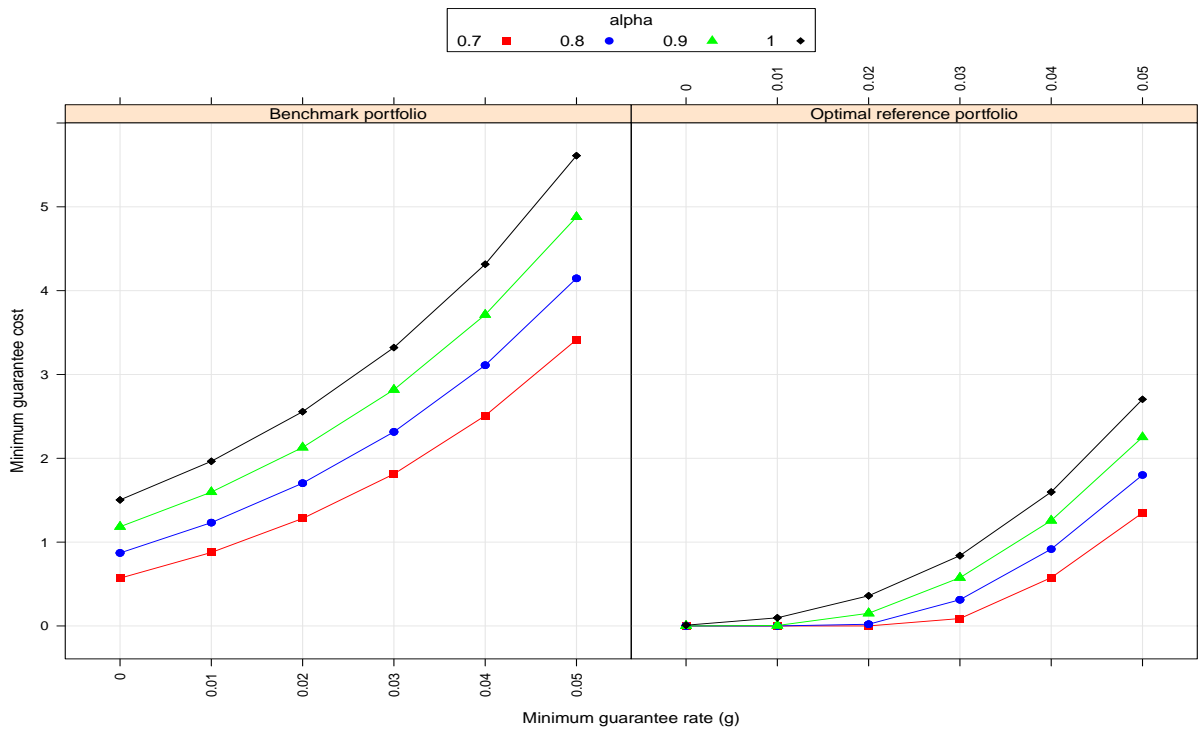


Figure 9: Benchmarking the cost of the guarantee of the State pension funds from Biggs (2011) against the optimised reference portfolio.

the figure that now the cost of the guarantee is 0.09, which can be charged to the employee. The pension fund will bear the option risk, but the cost is significantly lower than the previous risk sharing scheme. Of course, they have to correctly hedge the option but the advantage is clear: instead of paying 0.89 for each euro invested in the asset portfolio, there will be a cost of 0.39 (equity plus cost of the option).

We can also take the regulators' view. Let us assume that a pension fund pursues a very aggressive marketing policy (a kind of dumping) promising $g=3\%$. Then, the regulators should ask the pension fund to invest 0.3 from shareholders capital in the risky portfolio, and, at the same time, hedge the option. The latter means that in subsequent years the cost of the option must be covered by enough capital. Hence, the fund is "penalised" with a fair amount of capital requirement to account for the generous guarantee it promises.

5 Conclusions

This paper develops a general and computationally tractable model for pricing the cost of alternative embedded guarantee options in defined contribution pension funds. A particularly innovative and useful feature of the model is that it determines the asset allocation choice that is optimal for a given guarantee, in the sense that it minimizes the cost of the guarantee. The model is tested using real-world data to illustrate the effect of the design parameters of the guarantee on the cost of offering the option.

Results illustrate the effect on the option of (1) level of guarantee, (2) amount of equity, and (3) participation of the beneficiaries in any portfolio upswing over and above the

guarantee. We also show how the model can be used to benchmark existing portfolios and use it, in particular, to benchmark the average portfolios of State and local government pension funds studied in the literature. Our results are in agreement with the empirical findings of existing literature, but also attribute the precise cost of the guarantee. Finally we illustrate how the model can be used to calculate risk premia for risk sharing.

The model we implement and test can be extended to cover a broad range of guarantees that increasingly resemble DB, thus providing a continuum of funds in the hitherto dichotomous relation DC-DB. Some extensions are straightforward to build and test, while others are provided as areas for further research.

A Indices, volatilities and correlations

To generate the risk neutral paths, we compute volatilities and correlations of 12 indices, representing three broad classes: sovereign bonds, corporate bonds and stocks. We denote by `BONDS-1-3`, `BONDS-3-5`, `BONDS-5-7` and `BONDS-7-10` the J.P. Morgan aggregate indices of sovereign bonds issued by European countries with the indicated maturity ranges. The corporate bond classes are Salomon indices —`CORP-FIN`, `CORP-ENE` and `CORP-INS`— of bonds issued globally by financial, energy and insurance companies, respectively. Stock market indices are the Morgan Stanley Capital International Global, partitioned according to geo-political areas: EMU markets `STOCKS_EMU`, non-EMU `STOCKS-EX-EMU`, Pacific rim `STOCKS-PAC`, emerging economies `STOCKS-EMER`, and North-American `STOCKS-NA`. We estimate volatilities and correlations on 62 monthly observations, from Feb. 1995 to March 2000. Yearly volatilities are obtained by the square root rule.

Index Name	Volatility
<code>BONDS-1-3</code>	2.03E-02
<code>BONDS-3-5</code>	3.06E-02
<code>BONDS-5-7</code>	4.13E-02
<code>BONDS-7-10</code>	4.76E-02
<code>CORP-FIN</code>	4.26E-02
<code>CORP-ENE</code>	5.27E-02
<code>CORP-INS</code>	5.56E-02
<code>STOCKS_EMU</code>	1.89E-01
<code>STOCKS-EX-EMU</code>	1.47E-01
<code>STOCKS-PAC</code>	1.76E-01
<code>STOCKS-EMER</code>	2.29E-01
<code>STOCKS-NA</code>	1.35E-01

Table 1: Yearly volatilities.

	BND-1-3	BND-3-5	BND-5-7	BND-7-10	CRP-FIN	CRP-ENE	CRP-INS	STK_EMU	STK-EX-EMU	STK-PAC	STK-EMER	STK-NA
BND-1-3	1.0											
BND-3-5	9.37E-01	1.0										
BND-5-7	8.30E-01	9.65E-01	1.0									
BND-7-10	7.71E-01	9.29E-01	9.92E-01	1.0								
CRP-FIN	2.04E-01	1.42E-01	8.19E-02	6.42E-02	1.0							
CRP-ENE	1.84E-01	1.16E-01	5.05E-02	2.72E-02	9.73E-01	1.0						
CRP-INS	1.82E-01	1.16E-01	5.33E-02	3.09E-02	9.76E-01	9.71E-01	1.0					
STK_EMU	-2.11E-01	-2.29E-01	-2.14E-01	-1.98E-01	8.11E-03	-4.44E-03	3.15E-03	1.0				
STK-EX-EMU	-4.60E-02	-4.87E-02	-3.91E-02	-3.51E-02	2.03E-02	1.30E-02	-4.04E-03	8.75E-01	1.0			
STK-PAC	4.54E-02	-2.50E-02	-6.65E-02	-8.78E-02	-2.29E-02	1.22E-02	-6.75E-03	5.92E-01	5.97E-01	1.0		
STK-EMER	-2.22E-01	-2.82E-01	-3.04E-01	-3.17E-01	-5.15E-02	2.29E-02	-3.78E-02	6.70E-01	6.84E-01	6.59E-01	1.0	
STK-NA	-4.07E-02	-7.93E-02	-8.73E-02	-8.98E-02	2.61E-01	2.68E-01	2.45E-01	7.90E-01	8.07E-01	5.70E-01	6.61E-01	1.0

Table 2: Correlations

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