When Money Dies: The Dynamics of Speculative Hyperinflations*

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Abstract

How fast can a fiat money become valueless? In a continuous-time monetary economy with trading frictions and CRRA preferences, a fiat money under speculative hyperinflation dies in finite time. Its lifespan shrinks as distrust (measured by the distance to the highest equilibrium) increases and as liquidity needs become more frequent. Contrarily, high seller's market power, intense specialization, and active fiscal policies slow down the demise of money. If currencies compete, the times it takes for a speculative hyperinflation to trigger a dollarization, and for the full dollarization to be completed, vary with currency substitutability and rates of return.

JEL Classification: E40, E50

Keywords: hyperinflation, money, search, New Monetarism

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‹‹The masses of the people began to falter in their faith; and suspicion or doubt once started, once under headway against paper credit, or any other system of credit, is like an avalanche. You must control it, or you are buried beneath its waves.››

Stephen Dillaye, The Money and the Finances of the French Revolution of 1789.

1 Introduction

How fast can a fiat money become valueless? The *assignat* introduced in France at the onset of the French Revolution, in December 1789, and described by Levasseur (1894, p.179) as "the biggest experiment with paper money ever tried" provides one answer to this question. While the assignat was initially backed by land and subject to legal restrictions, it became a full-fledged fiat currency after the Reign of Terror ended in 1795 (Sargent and Velde, 1995, p.508). As the assignat lost its backing, "it went on depreciating and falling in value until it approached so near to worthless that a beggar in the street would hardly deign to accept one" (Dillaye, 1877, p.39). By the middle of 1796, as shown in Figure 1, "the assignats were nothing more than waste paper, repudiated by the State as they were rejected by commerce" (Levasseur, 1894, p.193).[1]

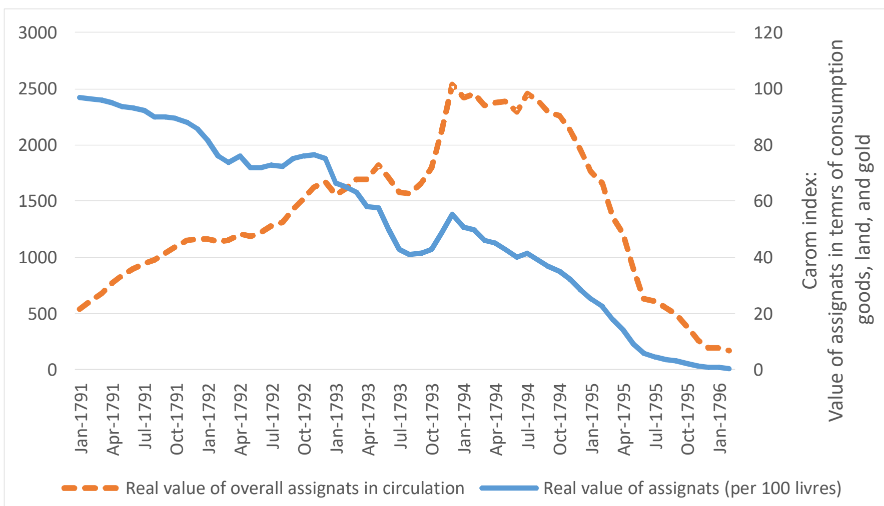


Figure 1: Plain curve: Real value of assignats per 100 livres measured according to the Caron index. See Footnote 35 in Sargent and Velde (1995). Dashed curve: Total quantity of assignats in circulation in real terms. Source: Aubin (1991), constructed from Tables A1 and A2.

The French example of the death of a fiat money is not unique.[2] The German Papiermark experienced hyperinflation in 1922-1923 with the monthly inflation rate peaking at 30,000 percent.[3] It was replaced with

[1] While the steep increase in the quantity of assignats in circulation was undoubtfully the main contributing factor of the demise of the assignat, other factors include widespread counterfeiting and speculation. See, e.g., Sargent and Velde (1995, p.508) and Stanziani (2011).

[2] The episode of the French assignat parallels the Continental currency introduced in 1775 to fund the U.S. revolutionary war. By the end of the war, in 1781, the real value of the Continental had been divided by a factor of 100. This episode gave rise to the idiom 'not worth a Continental,' which refers to anything worthless. See https://curiosity.lib.harvard.edu/american-currency/feature/continental-currency.

[3] For a description of the German hyperinflation, see Fergusson (1975).

the Rentenmark at the end of 1923. More recently, hyperinflation occurred in Zimbabwe, from March 2007 to November 2008, with a monthly inflation rate culminating at one hundred billion percent (Hanke and Kwok, 2009). In 2009, the Zimbabwe dollar was demonetized and the use of foreign currencies was legalized. These examples illustrate how a fiat money can disappear due to hyperinflation in a matter of months.

Models of fiat money provide laboratories to study hyperinflations. Typically, such models feature a continuum of equilibria where the value of money is positive at the initial date but converges to zero as time goes to infinity – a dire outcome called *speculative hyperinflation*. Despite the catastrophic welfare consequences associated with such equilibria, the literature studying them is narrow. When attention is given to speculative hyperinflations, it is primarily to eliminate such outcomes with equilibrium refinements, e.g., Wallace, (1981), Obstfeld and Rogoff (1983, 2021), Farmer (1984), and Nicolini (1996).

In this paper, I take the view that speculative hyperinflation equilibria are worthy objects of study as they embody the singularity of fiat money, i.e., speculative hyperinflations do not exist under a commodity-based monetary system. They capture the notion that expectations about the value of a fiat money can become unanchored from the "monetarist" value determined by the time-path of the money supply. As a result, the study of such equilibria can deepen our understanding of the real-world dynamics of inflations and hyperinflations.[4] This paper characterizes speculative hyperinflation equilibria analytically under different assumptions on market structure, trading mechanisms, and monetary and fiscal policy. It addresses topics such as the duration of a currency under speculative hyperinflation, the impact of market power on inflationary dynamics, the role of fiscal and monetary policy regimes, and the possibility for the economy to adopt a competing means of payment and dollarize.

Hyperinflations require high-frequency observations, e.g., at the peak of Hungary's hyperinflation in 1945-46, the daily inflation rate exceeded 200% (Bomberger and Makinen, 1983). Therefore, following the tradition of Cagan (1956) and Friedman (1971), I study economies where the demand for money is expressed in continuous time.[5] Relative to this earlier work, I consider environments where the role of money, and hence its demand, is microfounded. In order to establish some core results, I adopt first the search-theoretic model of Shi (1995) and Trejos and Wright (1995) – STW thereafter – which features indivisible money and a unit upper bound on individual money holdings. Later in the paper, I relax these restrictive assumptions.

I will show that under CRRA preferences and no bargaining power to sellers, the differential equation that governs the value of money, i.e., the amount of output that an indivisible unit of money can purchase, is an autonomous Bernoulli equation – analogous to the ODE of the Solow growth model – that admits closed-form solutions. The set of solutions includes a range of speculative hyperinflations indexed by the initial value of money within some open interval. Along these equilibria, money becomes worthless in finite time. The time it takes for money to lose its value is the product of two terms, the initial departure from the

[4] In Obsfeld and Rogoff's (2021) words: ‹‹Although apparently a narrow issue, studying these extreme economies turns out to be quite illuminating in understanding the fundamentals of price level determination.››

[5] For a description of the Hungarian hyperinflation of 1945-1946, see Bomberger and Makinen (1983).

monetary steady state, which is interpreted as a measure of the distrust in the future value of fiat money, and a duration term that is inversely related to the velocity of money and the rate of time preference. The velocity itself is endogenous and depends on trading frictions, the severity of the double-coincidence-of-wants problem, and the supply of money.

The result that fiat monies can have a finite horizon seems at odds with the following backward induction logic: in a deterministic equilibrium of a discrete-time economy, if money has no value at some date T then it cannot have a positive value at date $T-1$. Despite the apparent contradiction, I show that the equilibrium in continuous time is the limit of a sequence of equilibria in discrete time, where each of the equilibrium agrees with the backward induction logic above. I also show how to relate this result to a little-known property of the textbook Solow growth model regarding the convergence of the economy in backward time to its origin.

While the STW is highly tractable, it is limited in its ability to apprehend the monetary or fiscal views of hyperinflation. Therefore, I generalize the model to allow for divisible money, arbitrary paths for the money supply, and different trading mechanisms. It is done along the lines of Choi and Rocheteau (2021b), by adding a good that is traded competitively and enters preferences linearly, making the model a continuous-time version of the Lagos and Wright (2005) environment. The time-paths for aggregate real balances along all equilibria can be characterized in closed form. At the highest equilibrium, in accordance with the monetarist doctrine, the value of money is tied to the path of the money supply and remains bounded away from zero. In contrast, along speculative equilibria, fiat money loses its value in finite time and, for given initial conditions, the rate of divergence from the nonspeculative equilibrium increases with the money growth rate. Under a calibration that targets the semi-elasticity of US money demand, it takes between 70 and 115 years, depending on the frequency of liquidity needs, for fiat money to become valueless if it departs from its steady-state value by a mere one percent. If the monthly money growth rate exceeds 50 percent so that the economy enters an hyperinflation regime, it takes about 6 years for money to lose all its value if liquidity needs occur at a monthly frequency.

I explore the robustness of the results to several extensions, e.g., preferences with bounded marginal utility, trading mechanisms where sellers have bargaining power. I consider an extension where domestic currency competes with an alternative asset (e.g., foreign currencies or precious metals) for the role of means of payment. The theory determines how long an economy can remain undollarized and how long it takes for the economy to be fully dollarized. I also consider the case where money creation is endogenous and finances a constant flow of government purchases, as in Sargent and Wallace (1973).

Finally, I conclude the paper by showing that the phenomenon of speculative hyperinflation is not specific to pure currency economies. I describe an economy with pure credit under limited commitment, along the lines of Kehoe and Levine (1993), and show the existence of equilibria where debt limits in real terms converge to zero in finite time.

1.1 Literature review

This paper revisits an old question in monetary economics going back to Cagan (1956) – the dynamics of hyperinflations. The novelty consists in explaining how fast a fiat money dies along a *speculative* hyperinflation equilibrium depending on market structure and monetary policy. The methodology builds on the New Monetarist literature, surveyed in Lagos, Rocheteau, and Wright (2017), that provides microfoundations for the role of money by formalizing decentralized markets with trading frictions. The existence of speculative hyperinflation equilibria has been established in the context of continuous-time search-theoretic models by Trejos and Wright (1995, Section 6), Coles and Wright (1998), Trejos and Wright (2016), and Choi and Rocheteau (2021a,b), among others, and in discrete-time models by Lagos and Wright (2003) and Rocheteau and Wright (2013).[6] Speculative hyperinflations, however, are not the focus of these papers. To the best of my knowledge, my paper is the first to solve the entire time-paths of speculative hyperinflations for different trading mechanisms, different policy regimes, and different payment systems, and to establish that, under standard preferences, fiat money becomes valueless in finite time along a deterministic equilibrium.

From a methodology standpoint, the model has two key ingredients. First, the environment is written in continuous time, as in Shi (1995) and Trejos and Wright (1995), which is instrumental to obtain the (finite) duration of a fiat money. Second, money is perfectly divisible and individual asset holdings are unrestricted, as in Lagos and Wright (2003, 2005), which is necessary to analyze the role of monetary and fiscal policies for the dynamics of speculative hyperinflations. Relative to most of the literature, I study arbitrary time-paths for the money growth rate (and not simply constant money growth rates) and I endogenize these paths by linking them to the fiscal needs of the government.[7]

In this paper, I only consider deterministic, perfect foresight equilibria, which makes the disappearance of fiat money in finite time even more puzzling. Stochastic, sunspot equilibria are studied in Lagos and Wright (2003) and Rocheteau and Wang (2023) in a discrete-time and continuous-time environments, respectively. While there exist sunspot equilibria where fiat money becomes valueless with the realization of a sunspot state, those equilibria are qualitatively different from speculative hyperinflations. Along a sunspot equilibrium, the probability that money is alive at any time horizon is positive. Moreover, the probability of occurrence of the sunspot state that coordinates agents on the nonmonetary equilibrium is independent of fundamentals and policies. Finally, average inflation in stationary sunspot equilibria is determined solely by the (constant) money growth rate.

I study speculative hyperinflations under different policy regimes: when fiscal policy is passive and accommodates an exogenous time-path for the money supply and when fiscal policy is dominant and sets the size of the revenue from money creation in order to finance a constant flow of government purchases, as in Sargent and Wallace (1973). The seminal paper on the role of the coordination between monetary and

[6] Outside of the search literature, Brock (1975) provides the first study of perfect foresight equilibria, including speculative hyperinflation equilibria, for a dynamic, competitive economy with money in the utility function. A treatment of dynamic equilibria in overlapping generation economies is provided by Azariadis (1993).

[7] A noticeable exception is Gu, Han, and Wright (2020) who study news about monetary policy in a New Monetarist model.

fiscal policies for the dynamics of inflation is Sargent and Wallace (1981). Sargent (1982) emphasized the role of fiscal policy to explain the end of hyperinflation episodes. A thorough review of the fiscal theory of inflation is provided by Cochrane (2023). Relative to this literature, I focus on *speculative* hyperinflations, and not *fundamental* hyperinflations, and I discuss the links between the two.

I follow Sargent and Wallace (1973) by studying hyperinflation under rational expectations (more precisely, under perfect foresight). Alternatively, adaptive learning provides a realistic description of inflation expectations and can be used to refine the equilibrium set. Marcet and Sargent (1989) show that the high-inflation steady state in a model where the government uses money creation to finance government purchases is unstable under least-squares learning. Adam, Evans, Honkapohja (2006) reconsider this result under different informational assumptions and show that speculative paths are stable under least-squares learning if agents can utilize contemporaneous data.

The version of the model with a domestic currency and a competing asset or currency is related to Lester, Postlewaite and Wright (2012), Zhang (2014), and Gomis-Porqueras, Kam, and Waller (2017), among many other papers on dual currency economies. Altermatt, Iwasaki, and Wright (2023) study the sunspot and nonstationary equilibria of a discrete-time New Monetarist economy with multiple liquid assets.

The existence of speculative equilibria in the context of pure credit economies with limited commitment and endogenous debt limits is discussed in Bloise, Reichlin, and Tirelli (2013), Gu, Mattesini, Monnet, and Wright (2013), and Bethune, Hu, and Rocheteau (2018). In the last section of the paper, I solve these equilibria in closed form in a continuous-time environment and show that debt limits can converge to zero in finite time.

2 Hyperinflations in the STW model

Some results in this section will likely appear at odds with the common wisdom. Therefore, I derive them first in a textbook, microfounded model of a pure currency economy due to Shi (1995) and Trejos and Wright (1995). The environment builds on Diamond (1982) where all trades take place in pairwise meetings formed at random in continuous time, and on Rubinstein and Wolinsky (1985) where terms of trade are determined according to extensive-form bargaining games. Relative to Diamond (1982), agents are specialized in terms of production and consumption, which generates a double-coincidence-of-wants problem and, in the absence of a public record technology (Kocherlakota, 1998), a role for money. In Section 3, I will show the results hold in a generalized version of STW that is more suitable to discuss the link between hyperinflation and money growth and will consider applications and extensions.

2.1 The environment

Time is continuous. There is a unit measure of agents divided evenly into $N \geq 3$ types and N divisible commodities. An agent of type $n \in \{1, ..., N\}$ produces good n and consumes good $n+1$ (modulo N).[8] The utility of consuming $y \in \mathbb{R}_+$ units of a desired good is $u(y)$, with $u(0) = 0$, $u' > 0$, and $u'' < 0$. The disutility of producing y units of a commodity is simply y.[9] The rate of time preference is $\rho > 0$. Agents meet pairwise and at random according to a Poisson process with arrival rate $\alpha > 0$. Conditional on a meeting, an agent likes the good produced by her partner with probability $\sigma = 1/N$. The specialization in terms of production and tastes rules out the double coincidence of wants whereby the two agents in a match like each other's output. There is a quantity $M \in (0, 1)$ of fiat money, an intrinsically useless but durable object. For tractability (to reduce the state space), money is indivisible and agents can hold at most one unit.[10] Hence, the set of individual money holdings is simply $\{0, 1\}$. I will relax this assumption later. The terms of trade in pairwise meetings are determined by take-it-or-leave-it offers by buyers. I consider alternative bargaining solutions in a later section, in the generalized version of the model.

2.2 Definition of equilibrium

An equilibrium is composed of a pair of value functions solving Bellman equations and the solution to a bargaining problem. I start by characterizing agents' lifetime expected utilities. The value of an agent with one unit of money at time t, denoted by $V_{1,t}$, solves

$$\rho V_{1,t} = \alpha\sigma(1 - M) \left[u(y_t) + V_{0,t} - V_{1,t}\right] + \dot{V}_{1,t}, \tag{1}$$

where $V_{0,t}$ is the lifetime expected utility of an agent with zero unit of money. According to the right side of (1), the agent meets a producer of her consumption good at rate $\alpha\sigma$. This producer does not possess a unit of money, and hence can accumulate one unit if a trade takes place, with probability $1 - M$. In that event, the buyer enjoys the utility from consuming y_t units of goods, $u(y_t)$, in exchange for one unit of money, which corresponds to a loss in terms of lifetime expected utility equal to $V_{1,t} - V_{0,t}$. The last term on the right side is the change in the agents' lifetime utility over time. The value of an agent without money solves

$$\rho V_{0,t} = \alpha\sigma M \left(-y_t + V_{1,t} - V_{0,t}\right) + \dot{V}_{0,t}. \tag{2}$$

The interpretation is similar to the one of (1).

The output produced in exchange for one unit of money, y_t, is determined by take-it-or-leave-it offers by buyers. Hence, the buyer extends an offer that makes the seller indifferent between accepting or rejecting, which leads to

$$y_t = V_{1,t} - V_{0,t}. \tag{3}$$

[8] This pattern of specialization was formalized, e.g., in Kiyotaki and Wright (1989).

[9] In the STW model, there is no loss in generality in adopting a linear disutility of production, i.e., it is akin to a normalization consisting in choosing a unit for y.

[10] A version of the model with unrestricted money holdings is studied numerically by Molico (2006).

The left side is the disutility of production while the right side is the gain in terms of lifetime expected utility from receiving one unit of money. Substitute $V_{1,t} - V_{0,t}$ from (1)-(2) into the bargaining solution, (3), to obtain:

$$\dot{y}_t = [\rho + \alpha\sigma(1-M)]\, y_t - \alpha\sigma(1-M)u(y_t). \tag{4}$$

An equilibrium is a bounded solution, y_t, to the ODE (4). It says that the rate of appreciation of the value of money on the left side of (4) is equal to the opportunity cost of holding money, ρy, net of a liquidity premium, $\alpha\sigma(1-M)\,[u(y)-y]$.

For future use, I define the velocity of money, ϑ, as the aggregate value of all transactions per unit of time, $\alpha\sigma(1-M)M$, divided by the stock of money. Hence, it is equal to

$$\vartheta = \alpha\sigma(1-M). \tag{5}$$

It increases with the rate at which agents meet, α, and it decreases with specialization, N, and with the share of agents endowed with money, M.

2.3 CRRA preferences

The objective of this section is to characterize speculative hyperinflation equilibria in closed form. For this, I focus for now on constant-relative-risk-aversion (CRRA) utility functions, $u(y) = y^{1-\eta}$ with $\eta \in (0,1)$.[11] The ODE (4) takes the form of a Bernoulli equation

$$\dot{y}_t = [\rho + \alpha\sigma(1-M)]\, y_t - \alpha\sigma(1-M)y_t^{1-\eta}, \tag{6}$$

with a positive steady state given by

$$y^s = \left(\frac{\vartheta}{\rho+\vartheta}\right)^{\frac{1}{\eta}}. \tag{7}$$

There is also a nonmonetary steady state such that $y_t = 0$ for all t. The phase diagram of (6) is represented in the left panel of Figure 2. The monetary and nonmonetary steady states are determined at the intersection of the phase line and the horizontal line. For all $y_0 \in (0, y^s)$, there is a *speculative* hyperinflation trajectory starting at y_0 and converging to 0. The term *speculative* refers to the observation that along those equilibria, the change in the value of money is driven purely by self-fulfilling belief as the money supply and fundamentals are constant over time. By contrast to these speculative equilibria, I will refer to the highest monetary equilibrium, y^s, as the *nonspeculative* equilibrium.

We will encounter Bernoulli equations multiple times in this paper. Therefore, I solve (6) step by step. On any time interval, $\mathcal{T}$, such that $y_t > 0$, I adopt the change in variable $x_t = y_t^{\eta}$. Hence, $\dot{x}_t = \eta y_t^{\eta-1}\dot{y}_t$. I substitute $\dot{y}_t = \dot{x}_t y_t^{1-\eta}/\eta$ into (6) to obtain:

$$\dot{x}_t = [\rho + \alpha\sigma(1-M)]\,\eta x_t - \alpha\sigma(1-M)\eta, \quad \forall t \in \mathcal{T}. \tag{8}$$

[11] The coefficient η is restricted to be less than one in order to guarantee that the utility is bounded below, $u(0) = 0$, so that the surplus from trade is finite. I generalize the utility function later in Section 4.1.

This linear, first-order differential equation admits a positive steady state, $x^s = (y^s)^\eta$. Given an initial condition, $x_0 \in (0, x^s)$, the solution to (8) is

$$x_t = (x_0 - x^s)\, e^{[\rho+\alpha\sigma(1-M)]\eta t} + x^s, \quad \forall t \in \mathcal{T}. \tag{9}$$

Using that $y_t = x_t^{\frac{1}{\eta}}$, I recover the time-path for y_t. I compute the time at which money becomes valueless, T, by solving $x_T = 0$, which implies $y_T = 0$. Note from (6) that y_t is differentiable at $t = T$ with $\dot{y}_T = 0$. The results are summarized in the following proposition.

Proposition 1 ***(Speculative hyperinflations in the STW model.)*** *Suppose $u(y) = y^{1-\eta}$ with $\eta \in (0,1)$. There exists a continuum of speculative, hyperinflation equilibria indexed by $y_0 \in (0, y^s)$. For given $y_0 < y^s$, the time-path for the value of money is*

$$y_t = y^s \left\{ 1 - \left[\frac{(y^s)^\eta - y_0^\eta}{(y^s)^\eta} \right] e^{(\rho+\vartheta)\eta t} \right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t), \tag{10}$$

where $\mathbb{I}_{[0,T]}(t)$ is an indicator variable, $\vartheta = \alpha\sigma(1-M)$, and the time it takes for fiat money to lose its value is

$$T = \frac{\ln\left[1 - (y_0/y^s)^\eta\right]^{\frac{-1}{\eta}}}{\rho + \vartheta} < +\infty. \tag{11}$$

Proposition 1 characterizes the time-paths for the value of money along all speculative, hyperinflation equilibria in closed form. Each path depends on the initial value of money, y_0, preferences, ρ and η, idiosyncratic risk, α, specialization, σ, and the quantity of money, M.

The speculative component in (10) is

$$\overbrace{\left[\frac{(y^s)^\eta - y_0^\eta}{(y^s)^\eta} \right]}^{\text{initial gap}} \overbrace{e^{(\rho+\vartheta)\eta t}}^{\text{speed of divergence}}.$$

It is the product of two terms. The first one is the initial departure of the value of money from the steady state. The ratio, y_0/y^s, can be interpreted as a measure of agents' trust in fiat money. The second component is a growth factor that captures the rate at which the value of money drifts away from its nonspeculative value. This divergence rate is equal to $(\rho + \vartheta)\,\eta$.

In order to perform some comparative dynamics, I rewrite (10) as

$$\tilde{y}_t = \left[1 - (1 - \tilde{y}_0^\eta)\, e^{(\rho+\vartheta)\eta t} \right]^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t). \tag{12}$$

where $\tilde{y}_t \equiv y_t/y^s$. Consider the effects of reducing the trading frictions, i.e., $\alpha\sigma$ increases, taking $\tilde{y}_0$ as given. The speed of divergence, $[\rho + \alpha\sigma(1-M)]\,\eta$, increases. Hence, $\tilde{y}_t$ is lower for all $t > 0$. So while a reduction in trading frictions makes fiat money more valuable at the steady state, it makes it less valuable (relative to the steady state) along a speculative hyperinflation. I illustrate this result in the bottom panel of Figure 3 where the top horizontal line represents $y_t/y^s = 1$. The trajectory for $\tilde{y}_t$ is steeper at $\tilde{y}_0$ when $\alpha\sigma$ increases.

In the STW model, the quantity of money determines the composition of the market between buyers (agents with a positive payment capacity) and sellers (agents who are willing to produce). An increase in M reduces the share of sellers and hence the velocity of money. As a result, it reduces the speed of divergence along a speculative hyperinflation, as illustrated in the bottom panel of Figure 3.

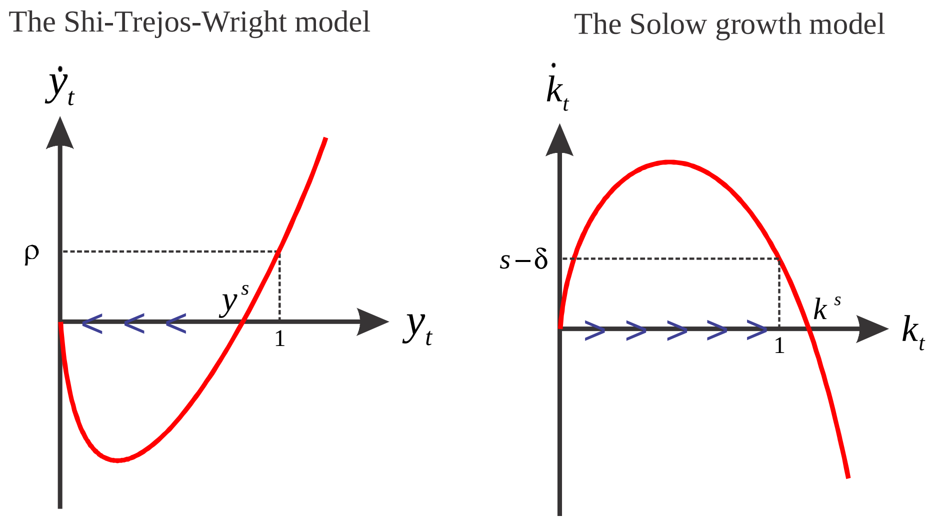


Figure 2: Phase diagrams of the Shi-Trejos-Wright and Solow growth models

So far, the speculative hyperinflation equilibria have been indexed by the initial value of money, y_0. Alternatively, they can be indexed by the time at which money becomes valueless.

Corollary 1 ***(Equilibria indexed by*** T***.)*** *There exists a continuum of monetary equilibria indexed by* $T \in (0, +\infty)$ *where*

$$y_t = y^s \left[1 - e^{-(\rho+\vartheta)\eta(T-t)}\right]^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t).$$

Corollary 1 shows that the STW model admits a continuum of monetary equilibria indexed by the time at which the value of fiat money vanishes. This result goes against two common wisdoms. First, there is the common wisdom in monetary theory that if fiat money is valued at any point in time along a perfect foresight equilibrium, then it should be valued at all dates. This wisdom is based on a simple backward induction logic. If money loses its value at time T, then it should not be accepted at time $T-1$ and hence its value should be zero at $T-1$. Why does this logic fail in continuous time? Because there is not one single instant before T, i.e., there is a continuum of dates between $T-\varepsilon$ and T for all $\varepsilon > 0$. So the backward induction logic cannot be invoked to argue that if $y_T = 0$ then $y_t = 0$ for all $t < T$. For any $T-\varepsilon$, agents are willing to hold onto their unit of money because there is a small time interval of length ε over which money can be spent, and the marginal utility of spending a unit of real balances is unbounded as $y_{T-\varepsilon}$ approaches zero.

The second common wisdom is that an out-of-steady-state solution to a dynamic system cannot converge to a steady state in finite time. If that were the case, the system would admit multiple solutions for the

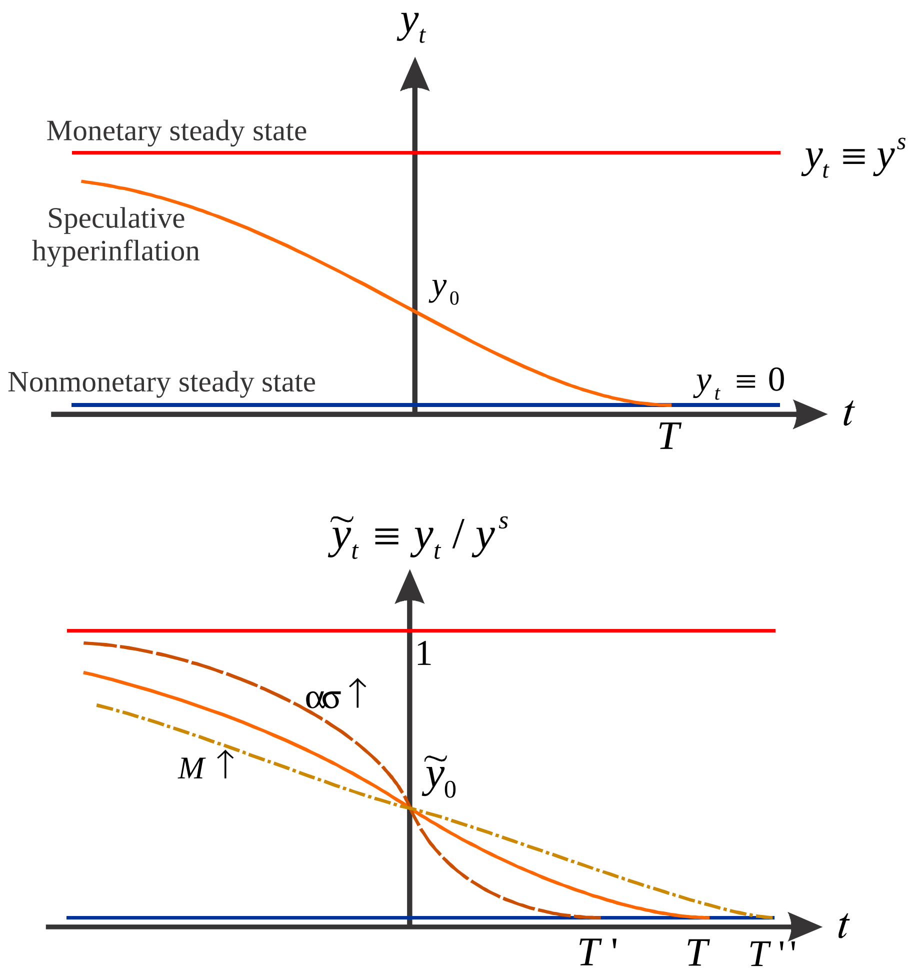


Figure 3: Top panel: Time-path solutions of the Shi-Trejos-Wright model. Bottom panel: Effects of an increase in $\alpha\sigma$ or M.

some initial conditions. (Recall that solutions are defined on the entire real line, $\mathbb{R}$, so that time can move forward or backward.) The multiplicity of solutions of the ODE of the STW model manifests itself when $y_T = 0$. As shown in the top panel of Figure 3, if $t > T$ and one moves backward in time, then there is a time path where $y_\tau = 0$ for all $\tau < t$ and there is another time path where $y_\tau > 0$ for all $\tau < T$. By the theorem of Cauchy-Lipschitz, a *sufficient* condition for uniqueness of a solution to an autonomous ODE is that $\dot{y}_t$ as a function of y_t is locally Lipschitz.[12] This condition is not satisfied in the neighborhood of 0 for CRRA utility functions since the marginal utility of consumption is unbounded.

[12] Consider an autonomous, differential equation $\dot{x}_t = f(x_t)$ where $x_t : \mathbb{R} \mapsto \mathbb{R}$. The function f is locally Lipschitz if there exists a constant $L > 0$ such that $|f(y) - f(z)| \leq L\,|y - z|$ for all y and z in a neighborhood of x, for all x. While this condition might seem complicated, it suffices that f is C^1.

2.4 Speculative hyperinflations in discrete time

The result that money can become valueless in finite time is counter-intuitive as it seems to violate a simple backward induction logic. I now show that the logic holds in the following sense: equilibria in continuous are the limits of equilibria in discrete time, for which backward induction holds, when the length of the period approaches zero. I derive in Appendix B the definition of an equilibrium of the STW model in discrete time as a sequence, $\{y_{n\Delta}\}_{n=0}^{+\infty}$, solution to

$$y_{n\Delta} = \beta_\Delta \left\{\alpha_\Delta \sigma (1-M) \left[u(y_{(n+1)\Delta}) - y_{(n+1)\Delta}\right] + y_{(n+1)\Delta}\right\}. \tag{13}$$

The length of a period corresponds to Δ, which affects both the discount factor, $\beta_\Delta = e^{-\rho\Delta}$, and the arrival rate of meetings, $\alpha_\Delta = 1 - e^{-\alpha\Delta}$. It can be checked that the difference equation (13) converges to the differential equation (4) as Δ approaches zero. To see this, denote $t = n\Delta$, and rearrange (13) as follows:

$$\left(\frac{1-\beta_\Delta}{\beta_\Delta}\right) y_t = \alpha_\Delta \sigma (1-M) \left[u(y_{t+\Delta}) - y_{t+\Delta}\right] + y_{t+\Delta} - y_t.$$

Divide by Δ and take the limit as $\Delta \to 0$ to obtain (4). In Figure 4, I plot equilibria for different values of Δ.

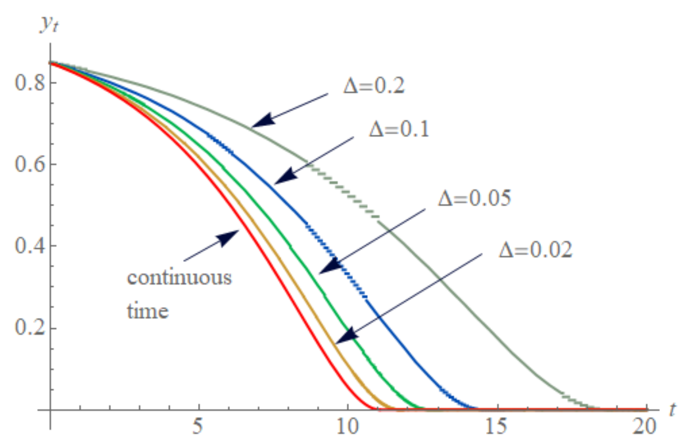


Figure 4: Comparison of equilibria in discrete time and the equilibrium in continuous time. $u(y) = \sqrt{y}$, $\rho = 0.01$, $\alpha = 5$, $\sigma = 0.2$, $M = 0.5$, and $y_0 = 0.85$.

One can see from Figure 4 that the equilibrium time-paths in discrete time converge to the equilibrium time-path in continuous time as Δ becomes small. Even though the solution to (13) is such that $y_{n\Delta} > 0$ for all n, the limit of this solution as Δ goes to zero is such that $\lim_{\Delta\to 0, \Delta n \to t} y_{n\Delta} = 0$ for all $t > T$.

2.5 A detour via the Solow growth model

The representation of the equilibrium of a macroeconomic model by a Bernoulli equation is not unique to the STW model. Indeed, the equilibrium of the textbook model of economic growth from Solow (1956) can also be represented by a Bernoulli equation. In the following, I show the analogy between the two equilibrium conditions and I establish a little-known result for the Solow growth model that mirrors the result from the STW model according to which money loses its value in finite time.

Under a Cobb-Douglas production function, the capital stock per worker obeys the following differential equation,

$$\dot{k}_t = sk_t^a - \delta k_t, \tag{14}$$

where $s \in (0,1)$ is the savings rate, $\delta > 0$ is the rate of depreciation. (Without loss, I omit technological progress and population growth.) The ODE is represented by a phase diagram in the right panel of Figure 2. The phase line is the mirror image of the phase line of the STW model relative to the horizontal axis. There is an active and an inactive steady state, the active steady state is dynamically stable. By the same logic as above, using the change of variable $x = k^{1-a}$, the closed form solution is

$$k_t = \left[\left(k_0^{1-a} - \frac{s}{\delta}\right) e^{-\delta(1-a)t} + \frac{s}{\delta}\right]^{\frac{1}{1-a}}.$$

As time goes to infinity, k_t approaches its steady state, $(s/\delta)^{1/(1-a)}$, asymptotically.[13] Given any $k_0 < (s/\delta)^{1/(1-a)}$, we can compute the finite time at which the economy started to grow:

$$T = \frac{-1}{\delta(1-a)} \ln\left(\frac{s}{s - \delta k_0^{1-a}}\right) > -\infty. \tag{15}$$

So, just like the steady state nonmonetary equilibrium is reached in finite time in the STW model, the inactive steady state is reached in finite "backward time" in the Solow model.

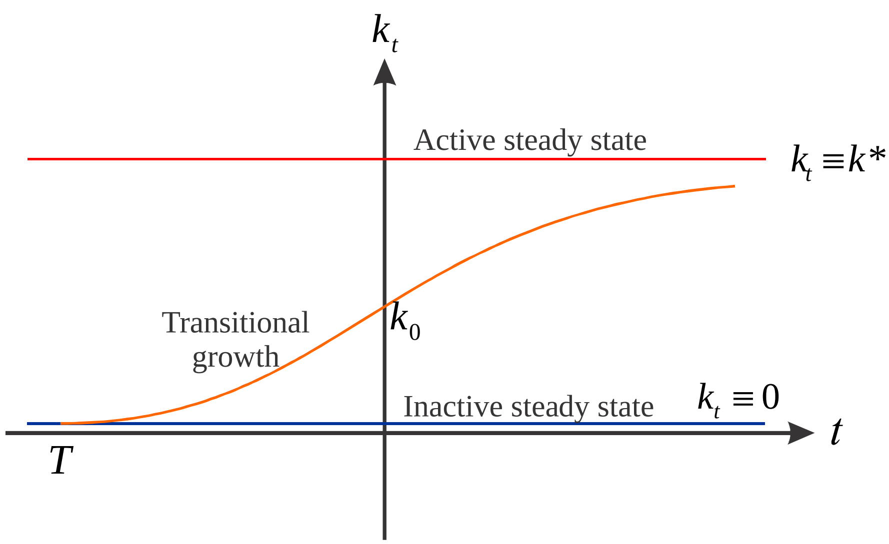


Figure 5: Solutions to the Solow growth model

Corollary 2 *Consider the Solow growth model and suppose $k_0 = 0$. There exists a continuum of equilibria indexed by $T \in \mathbb{R}_+$ such that the economy takes off at time T and reaches the active steady state asymptotically.*

[13] Sato (1963) is the first to identify the ODE of the Solow growth model under Cobb-Douglas production function as a Bernoulli equation and to provide a closed-form solution.

The solution to the equilibrium ODE of the Solow growth model is not unique when $k_0 = 0$. There is a solution where the capital stock remains at zero forever. But there are also a continuum of solutions indexed by the time at which the economy takes off.

How can the economy take off without capital? In the discrete-time Solow growth model, if $k_0 > 0$, then $k_{n\Delta} > 0$ for all $n \in \mathbb{Z}$ where Δ represents the length of a period. However, by the same reasoning as the one in Section 2.4, for all $n < T/\Delta$, where T is defined by (15), $k_{n\Delta}$ approaches 0 as Δ tends to 0. Capital is positive but infinitely small far away in the past. This infinitesimal capital combined with an infinite marginal product allows the economy to take off at some date T.[14]

3 The generalized model

The STW model adopts stark assumptions to remain analytically tractable, e.g., money is indivisible and there is a unit upper bound on individual money holdings. As a result, the model cannot study the main culprit for hyperinflation, namely money growth. In order to overcome this limitation, I generalize the STW model along the lines of Choi and Rocheteau (2021b) to have divisible money and arbitrary time-paths for the money growth rate.[15]

3.1 The environment

I now separate agents into two groups: a unit measure of buyers and a unit measure of sellers. As in STW, agents consume good y infrequently, at times T_n, $n = 1... + \infty$, following a Poisson process with arrival rate $\alpha\sigma > 0$. In addition, there is a perishable good that agents can consume and produce continuously through time and that can be traded competitively. I take this good as the numéraire. Buyers' preferences are represented by the following lifetime expected discounted utility:

$$\mathcal{U}^b = \mathbb{E}\left\{ \overbrace{\sum_{n=1}^{+\infty} e^{-\rho T_n} u\left[y(T_n)\right]}^{\text{STW preferences}} - \int_0^{\infty} e^{-\rho t} dH(t) \right\}, \tag{16}$$

where $H(t)$ is the cumulative production of the numéraire. (If $dH(t) < 0$, buyers consume the numéraire good).[16] Sellers' preferences are represented by

$$\mathcal{U}^s = \mathbb{E}\left\{ \int_0^{\infty} e^{-\rho t} dC(t) - \overbrace{\int_0^{+\infty} e^{-\rho t} dY(t)}^{\text{STW preferences}} \right\}, \tag{17}$$

[14] There is a parallel with the Friedmann equations in physical cosmological physics that govern the expansion of space. These equations show that the universe is of finite age, and had its origin in a mathematical singularity. See https://ned.ipac.caltech.edu/level5/Peacock/Peacock3_2.html

[15] This model can also be viewed as a continuous-time version of the model by Lagos and Wright (2005) and Rocheteau and Wright (2005) where centralized and decentralized markets open concurrently.

[16] In Rocheteau, Weill, and Wong (2018), $H(t)$ admits a density $h(t)$ with $h(t) \leq \bar{h} < +\infty$ for all t. In that case, all equilibria feature a nondegenerate distribution of money holdings. In order to obtain a degenerate distribution, I assume here that $H(t)$ can have a countable number of discontinuities.

where $C(t)$ is the cumulative consumption of the numéraire. The second integral represents the disutility of producing y where $Y(t)$ is the cumulative production.

Critically, at times $\{T_n\}_{n=1}^{+\infty}$, the buyer does not have access to the technology to produce the numéraire. Hence, she cannot pay for y with the numéraire good, which creates a need for money. The money growth rate is $\pi_t \equiv \dot{M}_t/M_t$ and the new money created is rebated to buyers lump-sum. These transfers (or taxes if $\pi < 0$), expressed in terms of the numéraire, are denoted ζ_t. For now, π_t is exogenous but I will consider later the case where π_t is endogenous and determined by the fiscal needs of the government. The price of money in terms of the numéraire is denoted ϕ_t. The rate of return of money is $r_t = \dot{\phi}_t/\phi_t$.

3.2 Definition of equilibrium

Consider a buyer with real asset holdings equal to m (expressed in the numéraire). The value function of a buyer at time t is linear in m, $V_t^b(m) = m + V_t^b$, where V_t^b solves the following HJB equation:

$$\rho V_t^b = \max_{m \geq 0} \left\{ -(\rho - r_t)m + \alpha\sigma v(m) + \zeta_t + \dot{V}_t^b \right\}, \tag{18}$$

where

$$v(m) = \max_{y \geq 0} \left\{ u(y) - p(y) \text{ s.t. } p(y) \leq m \right\}, \tag{19}$$

is the buyer's surplus from a bilateral trade, and where $p(y)$ is the total payment in terms of numéraire for y units of goods.[17] The payment function, $p(y)$, can take different forms corresponding to different pricing mechanisms. I consider mechanisms such that the buyer's surplus, $u(y) - p(y)$, is nondecreasing and concave. This restriction is satisfied, e.g., by the proportional bargaining solution.[18] At every point in time, the buyer chooses her (real) money holdings to maximize the right side of (18). The first term is the flow cost of holding money. It is the difference between the rate of time preference, ρ, and the expected rate of return of money, r_t, multiplied by the real money holdings, m. The second term is the expected surplus from a bilateral trade which, from (19), is equal to the difference between the utility of consumption, $u(y)$, and the payment, $p(y)$, subject to the feasibility condition that the payment does not exceed the buyer's real money balances, m. Finally, the last term is the change of the value function over time conditional on the state.

The first-order condition for the choice of asset holdings, assuming interiority, is

$$\rho - r_t = \rho - \frac{\dot{\phi}_t}{\phi_t} = \alpha\sigma \left[\frac{u'(y_t)}{p'(y_t)} - 1 \right]. \tag{20}$$

The left side of (20) is the cost of holding money. The right side of (20) is the expected liquidity value of a unit of real balances in terms of the buyer's surplus in pairwise meetings. From market clearing, $m_t = \phi_t M_t$. Hence, $\dot{m}_t/m_t = \pi_t + \dot{\phi}_t/\phi_t$. After substituting $r_t = \dot{m}_t/m_t - \pi_t$ into (20), I obtain the following ODE,

$$\rho + \pi_t - \frac{\dot{m}_t}{m_t} = \alpha\sigma L(m_t), \tag{21}$$

[17] Details about the derivation of this HJB equation are provided in Choi and Rocheteau (2021b).

[18] For a similar approach adopting a general description of pricing mechanisms in monetary economies, see Gu and Wright (1996).

where $L(m) \equiv u'[y(m)]/p'[y(m)] - 1$ is the marginal value to the buyer of one unit of real balances in a match and $y(m)$ is the solution to (19). An equilibrium is a bounded time path, m_t, that solves the ODE, (21).

3.3 Hyperinflation and money growth

I now show that the money growth rate plays an important role for the dynamics of speculative hyperinflations. For this, I solve for the entire set of perfect-foresight equilibria under a general time-path for the money growth rate, π_t. I adopt a simple pricing mechanism, $p(y) = y$, that leaves no surplus to sellers.[19] Preferences are of the CRRA form, $u(y) = y^{1-\eta}/(1-\eta)$, where $\eta \in (0, 1)$. (I discuss in Section 4.1 the case $\eta \geq 1$.)

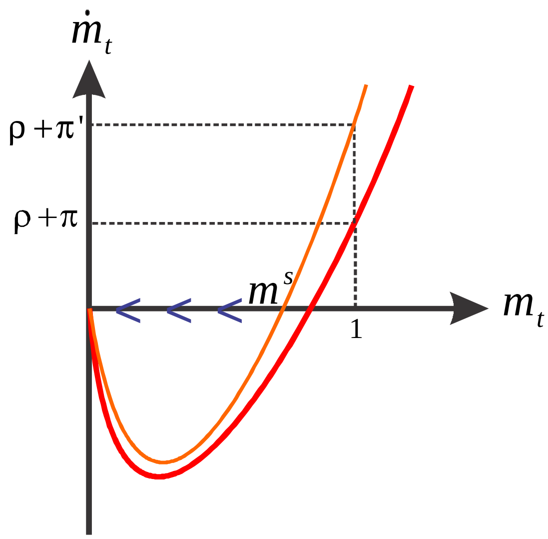


Figure 6: Phase diagrams when $u(y) = y^{1-\eta}/(1-\eta)$ with $\eta \in (0, 1)$.

I consider equilibria where the liquidity constraint binds at all dates, $m_t < y^*$. From (21), where $L(m_t) = m_t^{-\eta} - 1$, an equilibrium is a solution to

$$\dot{m}_t = (\alpha\sigma + \rho + \pi_t)m_t - \alpha\sigma m_t^{1-\eta}. \tag{22}$$

If the money growth rate is constant, $\pi_t \equiv \pi$, then the monetary steady state is

$$m^s = \left(\frac{\alpha\sigma}{\alpha\sigma + \rho + \pi}\right)^{\frac{1}{\eta}}. \tag{23}$$

From (23), $m^s > 0$ for all $\pi > 0$, i.e., money can be valued even at very high inflation rates. Real balances vanish, $m^s \to 0$, only if $\pi \to +\infty$. The ODE, (22), is represented in Figure 6 by a phase line that intersects the horizontal axis at the nonmonetary, $m = 0$, and monetary, $m = m^s$, steady states. There are a continuum

[19] This pricing function is consistent with competitive trades, e.g., as in overlapping generation models of fiat money (Wallace, 1980), or bilaterally trades with buyers possessing all the bargaining power. If the market for good y is competitive, the opportunities to consume good y at times T_n are interpreted as preference shocks.

of equilibria indexed by $m_0 \in (0, m^s)$ such that m_t converges to 0 at $t \to +\infty$. As the money growth rate increases, the phase line shifts upwards, which lowers m^s.

I turn to the general case where π_t is an arbitrary function of time. As before, (22) is solved by operating the change of variable, $x_t = m_t^\eta$. The resulting ODE is linear in x_t but it is not autonomous because π_t is allowed to vary over time. It is solved by using the method of the integrating factor.

Proposition 2 ***(Hyperinflations in competitive economies under CRRA preferences.)*** *Suppose $p(y) = y$ and $u(y) = y^{1-\eta}/(1-\eta)$ with $\eta \in (0,1)$. The time-path of the money growth rate is $\pi_t \in (-\rho, \bar{\pi})$, with $\bar{\pi} < +\infty$, and $\Pi(t) = \int_0^t \pi_s ds$ is the cumulative money growth rate. There exists a continuum of equilibria indexed by $T \in (0, +\infty]$ such that*

$$m_t = \left[\alpha\sigma\eta \int_t^T e^{-\eta\{(\alpha\sigma+\rho)(s-t)+[\Pi(s)-\Pi(t)]\}} ds\right]^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t). \tag{24}$$

If $T < +\infty$, money becomes valueless in finite time. If $T = +\infty$, the value of money is bounded away from zero at all times.

Proposition 2 derives the time-paths of aggregate real balances for all perfect-foresight equilibria where each equilibrium is indexed by the time at which money becomes valueless, T.[20] The nonspeculative equilibrium is defined as the equilibrium with the highest real balances at all points in time, which corresponds to $T = +\infty$. It is denoted $\bar{m}_t = \lim_{T\to\infty} m_t(T)$ for all t. From (24),

$$\bar{m}_t = \left[\alpha\sigma\eta \int_t^{+\infty} e^{-\eta\alpha\sigma(s-t)} e^{-\eta[\rho(s-t)+\Pi(s)-\Pi(t)]} ds\right]^{\frac{1}{\eta}} \quad \text{for all } t \geq 0. \tag{25}$$

In accordance with the monetarist doctrine, the value of money at time t depends negatively on the future cumulative growth of the money supply, $\Pi(s) - \Pi(t)$, at different time horizons, $s \geq t$. It is independent of past money growth rates. As a result, episodes of high inflation end when money growth rates are brought down. If the money growth rate is constant, $\Pi(t) = \pi t$, then $\bar{m}_t \equiv m^s$.

For all $T < +\infty$, there exists a speculative hyperinflation equilibrium such that money becomes valueless at time T. The solution (24) shows that the value of money depends on π_t for all $t < T$. However, money growth rates beyond T do not affect the value of money, even if the government commits to stop inflating the money supply.

The next corollary offers an alternative characterization of the equilibrium set, analogous to the one in Proposition 1, where each equilibrium is indexed by the value of real money balances at time $t = 0$, m_0.

Corollary 3 *For all $m_0 \in (0, \bar{m}_0)$, there exists a speculative hyperinflation equilibrium. It is such that:*

$$m_t = \left\{(\bar{m}_t)^\eta - e^{\eta[(\alpha\sigma+\rho)t+\Pi(t)]}\left[(\bar{m}_0)^\eta - (m_0)^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t), \tag{26}$$

where T solves

[20] One can interpret $\Pi(t)$ as a policy announcement of the path of future money growth rates. For a study of policy announcements in the context of the Lagos-Wright model, see Gu, Han, Wright (2020).

$$(\alpha\sigma+\rho)\,T+\Pi(T)=\ln\left[\frac{(\bar{m}_0)^\eta-(m_0)^\eta}{(\bar{m}_T)^\eta}\right]^{\frac{-1}{\eta}}. \tag{27}$$

From (26), $(m_t)^\eta$ can be decomposed into a nonspeculative component, $(\bar{m}_t)^\eta$, and a speculative component,

$$(\bar{m}_t)^\eta-(m_t)^\eta=\left[(\bar{m}_0)^\eta-(m_0)^\eta\right]e^{\eta[(\alpha\sigma+\rho)t+\Pi(t)]}. \tag{28}$$

The first term, $(\bar{m}_0)^\eta-(m_0)^\eta$, represents the initial unanchoring of real balances from their fundamental (nonspeculative) value. This speculative bias grows exponentially over time, where the growth rate is given by the velocity of money, $\alpha\sigma$, the rate of time preference, ρ, and the money growth rate, π_t. The role of the money supply is explained as follows. Dividing (26) by M_t, the value of money at time t is given by

$$\phi_t=\left\{\left(\bar{\phi}_t\right)^\eta-e^{\eta(\alpha\sigma+\rho)t}\left[\left(\bar{\phi}_0\right)^\eta-(\phi_0)^\eta\right]\right\}^{\frac{1}{\eta}}\mathbb{I}_{[0,T]}(t),$$

where $\bar{\phi}_t$ is the value along the nonspeculative equilibrium. So, each unit of money carries a negative bubble component, $(\phi_t)^\eta-\left(\bar{\phi}_t\right)^\eta$, that keeps growing over time at rate $\eta\,(\alpha\sigma+\rho)$. Since all units of money have the same price and the money supply grows at rate π_t, the aggregate speculative bubble grows at rate $\eta\,(\alpha\sigma+\rho+\pi_t)$.

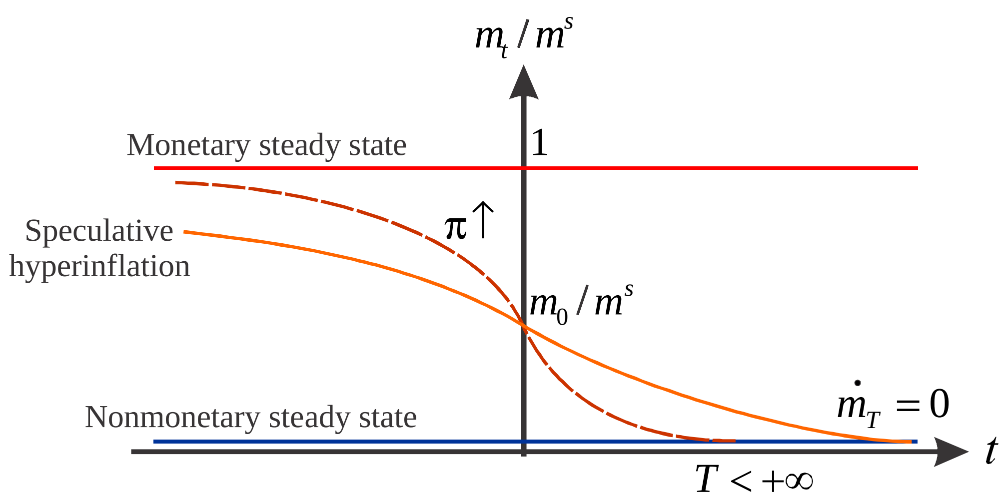


Figure 7: Time paths for aggregate real balances when $p(y)=y$ and $u(y)=y^{1-\eta}/(1-\eta)$ with $\eta<1$.

The time-paths for real balances are illustrated in Figure 7 for the case where $\pi_t\equiv\pi$. The red horizontal line corresponds to the monetary steady state while the blue horizontal line is the nonmonetary steady state. Speculative hyperinflation equilibria are between these two horizontal lines. I represent by an orange curve an equilibrium where the initial value of real balances normalized by the steady-state value is $m_0/m^s\in(0,1)$. As t tends to $-\infty$, m_t/m^s approaches one asymptotically. In contrast, moving forward in time, m_t/m^s reaches 0 at time

$$T=\frac{\ln\left[1-(m_0/m^s)^\eta\right]^{-1}}{(\rho+\alpha\sigma+\pi)\,\eta}<+\infty. \tag{29}$$

As π increases, the time-path for the speculative hyperinflation becomes steeper when going through the same (normalized) initial condition, m_0/m^s, and it reaches the nonmonetary steady state at $T' < T$.

3.4 Quantitative implications

I now provide a quantitative answer to the question that motivated this paper: How fast can a fiat money become valueless? I calibrate the model to the US economy following Lagos and Wright (2005). From (26), the key parameters for the dynamics of hyperinflation are α, σ, η, and π. The unit of time is a year so that the rate of time preference is set to $\rho = 0.04$. The annual rate of growth of the money supply is constant and equal to the average inflation rate, 4 percent. From (23), the semi-elasticity of money demand with respect to the nominal interest rate, $i \equiv \rho + \pi$, is

$$\left| \frac{\partial \ln m^s}{\partial i} \right| = \frac{1}{\eta \left(\alpha\sigma + \rho + \pi \right)}. \tag{30}$$

In the calibration of Lagos and Wright (2005), $|\partial \ln m^s / \partial i| \approx 10$, which is also the value used by Friedman (1971, p.851).[21] The same elasticity can be obtained from different combinations of η and $\alpha\sigma$, i.e., for any pair, (α, σ), there is an η that generates the elasticity of money demand observed in the data. Hence, the model is under-identified. In the following, I consider three different values for $\alpha\sigma$, $\{1, 10, 100\}$, corresponding to different frequencies of liquidity needs. If $\alpha\sigma = 1$, then liquidity needs occur once a year on average. If $\alpha\sigma = 10$, the frequency is close to a month. And if $\alpha\sigma = 100$, the frequency is bi-weekly. For each of these values, I set η so as to generate $|\partial \ln m^s / \partial i| = 10$. It can be seen in the table below that the curvature of $u(y)$ decreases as $\alpha\sigma$ increases.

Frequency of liquidity needs	Coefficient of RRA
$\alpha\sigma = 1$	$\eta = 0.093$
$\alpha\sigma = 10$	$\eta = 0.01$
$\alpha\sigma = 100$	$\eta = 0.001$

Figure 8 plots the value of T for these calibrated parameter values. In the top left panel, if m_0 is 99 percent of its steady-state value, it takes about 70 years for money to be valueless when $\alpha\sigma = 1$. If m_0 is 90 percent of m^s, then T falls to 46 years. So, the trust in the value of fiat money, as measured by m_0/m^s, plays an important role for the duration of fiat money. If the frequency of liquidity needs is higher, $\alpha\sigma = 10$ or $\alpha\sigma = 100$, then $T = 91$ and $T = 115$ years, respectively when $m_0/m^s = 0.99$. So, keeping the elasticity of money demand constant, fiat money is longer lived when the frequency of liquidity needs increases.

In the bottom panel, I plot T as a function of the monthly money growth rate for the different values of $\alpha\sigma$. I set m_0/m^s to 99%. The frequency of liquidity needs plays a critical role to explain the life expectancy of a fiat money in an hyperinflation regime, i.e., when π is larger 50% monthly. If $\alpha\sigma = 1$ and the monthly money growth rate is 50%, then $T = 0.6$, i.e., it takes 7 months for money to die. If $\alpha\sigma = 10$, then $T = 6.6$

[21] In the Lagos-Wright model, an agent is a buyer with probability σ, a seller with probability σ, and neither with probability $1 - 2\sigma$. Hence, $\sigma \leq 1/2$. In the calibration, σ is set at its highest value, 1/2. The meeting probability is set to $\alpha = 1$. The coefficient of RRA is chosen to minimize the distance between the model money demand and the money demand in the data. The value from this calibration is $\eta = 0.16$.

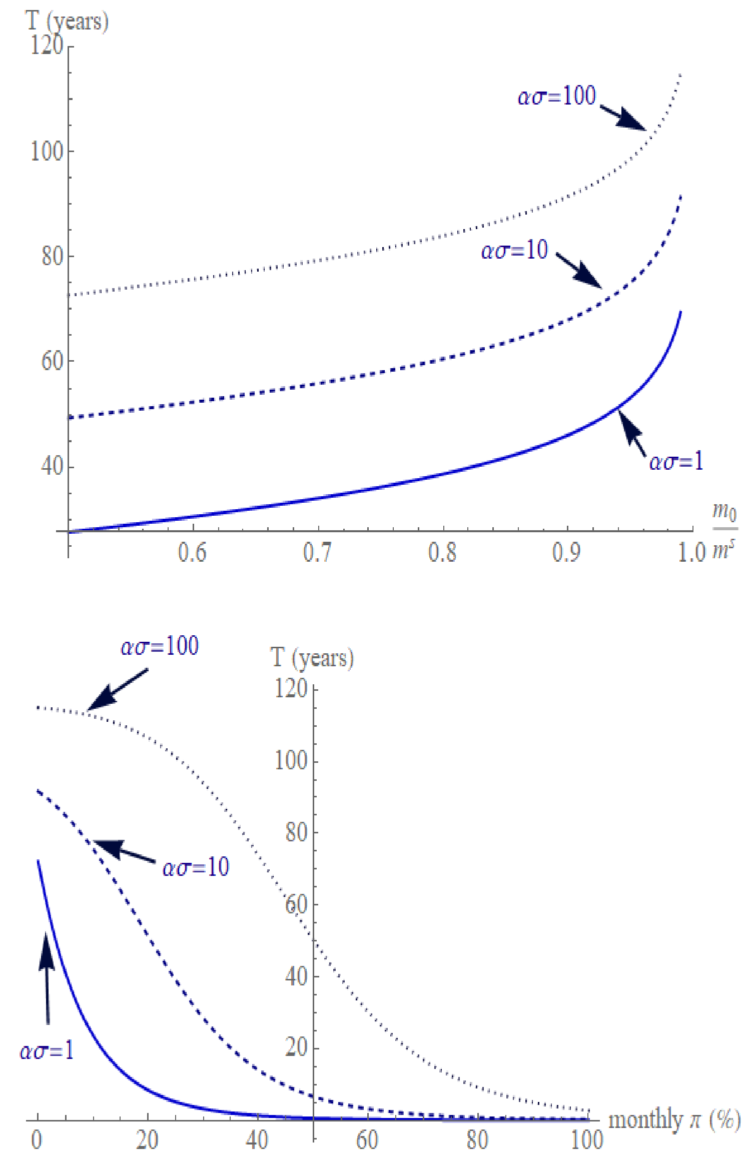


Figure 8: Time for fiat money to become valueless. Baseline parameter values: $\rho = 0.04$, $\pi = 0.04$, $\alpha\sigma \in \{1, 10, 100\}$, η is chosen such that $|\partial \ln m^s/\partial i| = 10$, and $m_0/m^s = 0.99$

years. If $\alpha\sigma = 100$, then $T = 50$ years. The duration of fiat money falls rapidly as the money growth rate increases. For instance, if the monthly money growth rate is 100% and $\alpha\sigma = 10$, then T is less than 3 months.

4 Extensions

I now discuss the role of some key assumptions. First, I will study utility functions that feature bounded marginal utility, e.g., modified CRRA preferences and quadratic preferences. Second, I will introduce bargaining solutions that give seller's market power.

4.1 When marginal utility is bounded

A key feature of the CRRA utility function is that the marginal utility of consumption is unbounded as consumption vanishes, $u'(0) = +\infty$. I now explore the dynamics of speculative hyperinflations when the marginal utility is bounded, $u'(0) < +\infty$. We will see how standard theorems for the existence and uniqueness of the solution to an ODE with initial condition can be applied. I will also show how CRRA preferences can be modified slightly in order to obtain speculative hyperinflations when the coefficient of relative risk aversion is greater than one.

Proposition 3 ***(Hyperinflations when marginal utility is bounded.)*** *Assume $\pi_t \equiv \pi$ and $p(y) = y$. The utility function, $u : \mathbb{Y} \to \mathbb{R}$, where $\mathbb{Y} \supset \mathbb{R}_+$ is an open and connected set, is C^∞. If $u'(0) > (\alpha\sigma + \rho + \pi)/\alpha\sigma$, then there exists a unique positive steady state, m^s, solution to*

$$u'(m^s) = 1 + \frac{\rho + \pi}{\alpha\sigma}. \tag{31}$$

There exist a continuum of speculative, hyperinflation equilibria indexed by $m_0 \in (0, m^s)$ featuring $m_t > 0$ and $\dot{m}_t < 0$ for all $t > 0$ with $m_t \to 0$ as time goes to $+\infty$. Moreover,

$$m_t = m^s - (m^s - m_0)\, e^{(\alpha\sigma+\rho+\pi)\eta(m^s)t} \quad \text{if } m_t \approx m^s \text{ and } t \approx 0, \tag{32}$$

$$m_t = m_0 e^{-\left[\alpha\sigma u'(0) - (\rho+\pi+\alpha\sigma)\right]t} \quad \text{if } m_0 \approx 0, \tag{33}$$

where $\eta(m) \equiv -mu''(m)/u'(m)$.

In Proposition 3, I assume u is defined on an interval that includes a neighborhood of 0 and it is infinitely differentiable. In particular, $u'(0) < +\infty$ and $u''(0) > -\infty$. Along a speculative hyperinflation equilibrium, when the marginal utility is bounded, fiat money has a positive value at all dates. The value of money approaches zero only at the limit when $t \to +\infty$. The expressions for m_t in (32) and (33) come from the linearization of the ODE in the neighborhood of $m_t = 0$ and $m_t = m^s$. When real balances are close to m^s, the rate of divergence from the steady-state monetary equilibrium is equal to the inverse of the semi-elasticity of money demand with respect to the rate of return of money,

$$\epsilon(m^s) \equiv \frac{\partial m^s / m^s}{\partial r} = \frac{1}{\alpha\sigma u'(m^s)\eta(m^s)}, \tag{34}$$

where, from (31), $\alpha\sigma u'(m^s) = \alpha\sigma + \rho + \pi$. When m_t is close to 0, the rate of convergence to the nonmonetary steady state, $\alpha\sigma u'(0) - (\rho + \pi + \alpha\sigma) > 0$, increases with $u'(0)$ and grows unbounded as $u'(0)$ tends to $+\infty$.

Generalized CRRA preferences A class of utility functions that satisfy the requirements in Proposition 3 while generalizing CRRA preferences is

$$u(y) = \frac{(y+b)^{1-\eta} - b^{1-\eta}}{1-\eta} \text{ with } b \geq 0. \tag{35}$$

When $b = 0$, the utility function corresponds to CRRA preferences. If $b = 0$ and $\eta \geq 1$, there is no speculative hyperinflation equilibrium. I first explain this non-existence result and then show it is not robust to small perturbations of b. The non-existence is more easily seen in the discrete-time model (e.g., Lagos and Wright, 2003) where, under CRRA preferences, real balances solve the following difference equation:

$$m_t = e^{-\rho}\left[(1-\alpha\sigma)\, m_{t+1} + \alpha\sigma (m_{t+1})^{1-\eta}\right].$$

It says that the value of money at time t, m_t, is equal to the discounted value at time $t+1$, $e^{-\rho} m_{t+1}$, multiplied by a liquidity premium factor, $1 + \alpha\sigma\left[(m_{t+1})^{-\eta} - 1\right]$. A sequence, $\{m_t\}_{t=0}^{+\infty}$, converges to 0 if as $m_{t+1} \to 0$ the solution m_t to the equation above also converges to 0. However, if $\eta > 1$, the right side goes to $+\infty$ as $m_{t+1} \to 0$. In continuous time, the nonexistence of speculative hyperinflation equilibria manifests itself as follows: from (22), $\dot{m}_{T^-} < 0$ while $\dot{m}_{T^+} = 0$, and hence m_t is not differentiable at T.

I show in the next corollary that speculative hyperinflations exist when $\eta \geq 1$ provided b is positive and small. I will consider the limit as b approaches zero and show the speculative hyperinflations converge to the ones described in Corollary 3.

Corollary 4 ***(Generalized CRRA preferences.)*** *Consider a sequence of utility functions, $\{u_n\}_{n=0}^{+\infty}$, defined by*

$$u_n(y) = \frac{(y+b_n)^{1-\eta} - (b_n)^{1-\eta}}{1-\eta},$$

with $\eta \geq 1$ and $b_n \in \left(0, [\alpha\sigma/(\alpha\sigma+\rho+\pi)]^{\frac{1}{\eta}}\right)$ for all $n \in \mathbb{N}_0$. Moreover, $\{b_n\}_{n=0}^{+\infty}$ is decreasing with $\lim_{n\to+\infty} b_n = 0$.

1. *For all $n \in \mathbb{N}_0$, there is a unique positive steady state,*

$$m_n^s = \left(\frac{\alpha\sigma}{\rho+\pi+\alpha\sigma}\right)^{\frac{1}{\eta}} - b_n. \tag{36}$$

2. *For all $n \in \mathbb{N}_0$ and all $m_0 \in (0, m_0^s)$, there exists a unique solution to (22), $m_{t,n} : \mathbb{R} \to [0, m_n^s]$. It is such that $m_{t,n} > 0$ and $\dot{m}_{t,n} < 0$ for all $t \in \mathbb{R}$ with $\lim_{t\to-\infty} m_{t,n} = m_n^s$ and $\lim_{t\to+\infty} m_{t,n} = 0$.*

3. *As $n \to +\infty$, $m_{t,n} : [0,T] \to [0, m_n^s]$ converges pointwise to*

$$\tilde{m}_t = \left\{(m_\infty^s)^\eta - e^{\eta(\alpha\sigma+\rho\pi)t}\left[(m_\infty^s)^\eta - (m_0)^\eta\right]\right\}^{\frac{1}{\eta}} \text{ for all } t \in [0,T], \tag{37}$$

where $m_\infty^s = [\alpha\sigma/(\rho+\pi+\alpha\sigma)]^{\frac{1}{\eta}}$ and

$$T = \frac{1}{\alpha\sigma + \rho + \pi} \ln \left[\frac{(m_\infty^s)^\eta - (m_0)^\eta}{(m_\infty^s)^\eta} \right]^{\frac{-1}{\eta}}. \tag{38}$$

Corollary 4 shows that even though a speculative hyperinflation does not exists when $b = 0$ and $\eta \geq 1$, i.e., RRA is greater than one, it does exist for b arbitrarily small and it approaches the solution derived in Section 3 for $\eta < 1$. This convergence result is illustrated in Figure 9 by plotting the numerical solution to (78) when $m_0 = 0.8$ for $\eta = 2$ and $b \in \{0.01, 0.05, 0.1\}$ and the solution at the limit when $b = 0$ given by (37). The lowest red plain line, when $b = 0$, is not differentiable at $m_t = 0$. All the other upper lines corresponding to $b > 0$ are differentiable at $m_t = 0$ with $\dot{m}_t = 0$.

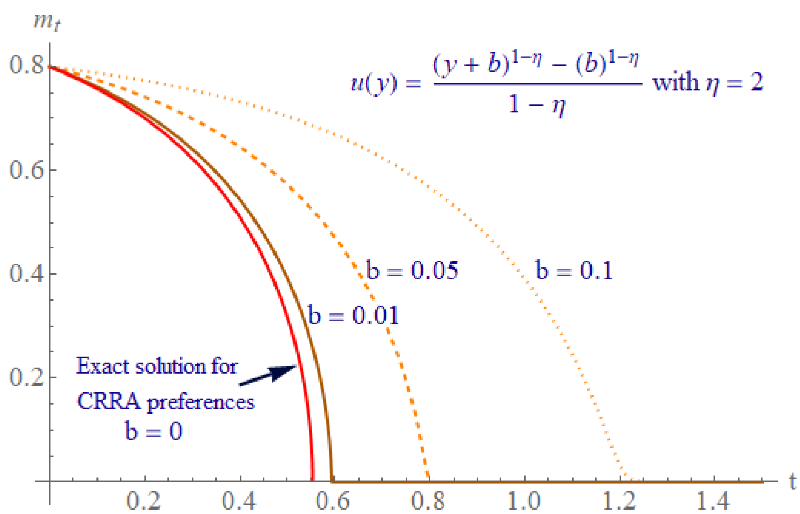


Figure 9: Generalized CRRA preferences with $\eta = 2$ and $b \in \{0, 0.001, 0.005, 0.1\}$.

Quadratic preferences Finally, I study a class of utility functions with bounded marginal utility for which the time-path for m_t can be solved in closed form. It will allow me to establish some similarities with the case of CRRA preferences studied earlier.

Proposition 4 ***(Speculative hyperinflations under quadratic preferences.)*** *Assume $u(y) = Ay - \varepsilon y^2/2$, $A > 0$, and $\varepsilon > 0$. If*

$$A > 1 + \frac{\rho + \pi}{\alpha\sigma}, \tag{39}$$

there exists a steady-state monetary equilibrium with

$$m^s = \frac{\alpha\sigma(A-1) - \rho - \pi}{\alpha\sigma\varepsilon}. \tag{40}$$

There also exists a continuum of speculative hyperinflation equilibria indexed by $m_0 \in (0, m^s)$ such that

$$m_t = \frac{m^s}{\underbrace{1 + \left(\frac{m^s - m_0}{m_0}\right) e^{[\alpha\sigma(A-1)-\rho-\pi]t}}_{speculative\ component}} \text{ for all } t > 0. \tag{41}$$

The speculative component in the expression for m_t in (41) appears at the denominator of m_t. When the condition (39) for the existence of a steady-state monetary equilibrium holds, the argument of the exponential function, $\alpha\sigma(A-1)-\rho-\pi$, is positive. So, as t goes to $+\infty$, the speculative component grows unbounded and m_t approaches zero asymptotically. For a given ratio, m^s/m_0, the speculative component at the denominator of (41) increases at a faster pace when liquidity needs, $\alpha\sigma(A-1)$, are higher.

A higher money growth rate, π, reduces the rate at which m_t diverges from its nonspeculative value, m^s. This result can be explained as follows. As established by (32), the rate of divergence from the positive steady state is given by the inverse of the elasticity of money demand. In the neighborhood of m^s, the relative risk aversion under quadratic preferences is $\eta(m^s)=\varepsilon m^s/(A-\varepsilon m^s)$ and the semi-elasticity of money demand is $\epsilon(m^s)=[\alpha\sigma(A-1)-\rho-\pi]^{-1}$. Money demand is more elastic at higher inflation rates reduces the rate at which m_t diverges from m^s.

4.2 Sellers' market power

I now illustrate the role of sellers' market power for the dynamics of hyperinflations by considering two alternatives bargaining solutions that generate variable markups.

Proportional bargaining The first bargaining solution, commonly used in the literature, is the Kalai proportional solution according to which the buyer's surplus in a pairwise meeting, $u(y)-p(y)$, is a constant share, $\theta\in(0,1)$, of the whole match surplus, $u(y)-y$, i.e.,

$$p(y)=\theta y+(1-\theta)u(y), \tag{42}$$

At the margin, the price of a unit of output is $p'(y)=1+(1-\theta)\left[u'(y)-1\right]$ where the second term is a markup over the marginal cost. If $u'(0)=+\infty$, e.g., under CRRA preferences, as y tends to zero, $p'(y)$ becomes unbounded.[22]

Proposition 5 ***(Speculative hyperinflations and market power: Proportional bargaining.)*** *Suppose $u(y)=y^{1-\eta}/(1-\eta)$ for $\eta\in(0,1)$, $\pi_t\equiv\pi$, and $p(y)$ is given by (42). If $\alpha\sigma>(\rho+\pi+\alpha\sigma)(1-\theta)$, then there exists a unique, positive steady state,*

$$m^s=\left[\frac{\alpha\sigma-(\rho+\pi+\alpha\sigma)(1-\theta)}{(\rho+\pi+\alpha\sigma)\,\theta}\right]^{\frac{1}{\eta}}, \tag{43}$$

and a continuum of equilibria, indexed by $m_0\in(0,m^s)$, such that $m_t>0$ for all $t>0$ with $m_t\to 0$ as time goes to $+\infty$. If m_0 is in the positive neighborhood of 0, then

$$m_t=m_0e^{-\left[\frac{\alpha\sigma\theta}{1-\theta}-(\rho+\pi)\right]t}. \tag{44}$$

[22]A thorough discussion of the properties of the Kalai bargaining solution in the context of monetary models is provided by Aruoba et al. (2007). Strategic foundations are given by Hu and Rocheteau (2020).

If $\theta < 1$, then, for all equilibria such that $m_0 \in (0, m^s)$, the value of money is positive for all $t \geq 0$. It converges to 0 only asymptotically. Indeed, even though $u'(y)$ grows unbounded as y_t goes to zero, so does $p'(y_t)$, and the ratio of the two stays finite. From (44), the rate at which the economy converges to the nonmonetary equilibrium, $\alpha\sigma\theta/(1-\theta) - (\rho + \pi)$, increases with θ. So, sellers' market power reduces the pace of speculative hyperinflations.

Gradual Nash bargaining The Nash solution is the most well-known axiomatic solution. However, in the context of models with liquidity constraints, it has undesirable properties (see Aruoba, Rocheteau, and Waller, 2007). So, I consider here a variant called the gradual (or ordinal) Nash solution. According to this solution, agents bargain sequentially over the output price of each unit of money held by the buyer. This approach is in spirit with the STW model where agents bargain over the amount of output a unit of money can buy. Heuristically, at the margin, the buyer and the seller negotiate the ∂y units of output and the payment, ∂p, subject to the constraint that $\partial p \leq \partial m$, where ∂m is small.[23] According to the generalized Nash solution,

$$(\partial y, \partial p) = \arg\max \left[u'(y)\partial y - \partial p\right]^{\theta} \left[\partial p - \partial y\right]^{1-\theta} \text{ s.t. } \partial p \leq \partial m,$$

where $u'(y)$ is the marginal utility of consumption given that the buyer has already purchased y in the previous rounds of the negotiation. From the first-order condition, the price of one unit of output is

$$p'(y) \equiv \frac{\partial p}{\partial y} = \frac{u'(y)}{\theta u'(y) + 1 - \theta} \quad \text{for all } y < y^*. \tag{45}$$

The difference relative to the proportional solution is that the price of a marginal unit of output is now bounded, $p'(0) = 1/\theta < +\infty$. Substituting $p'(y)$ from (45) into (21), the ODE for m_t is identical to (22) where $\alpha\sigma$ has been replaced with $\alpha\sigma\theta$.

Corollary 5 ***(Speculative hyperinflations and market power: Gradual Nash bargaining.)*** *Suppose $u(y) = y^{1-\eta}/(1-\eta)$ for $\eta \in (0,1)$, $\pi_t \equiv \pi$, and $p'(y)$ is given by (45). There exist a unique positive steady state,*

$$m^s = \left(\frac{\alpha\sigma\theta}{\rho + \pi + \alpha\sigma\theta}\right)^{\frac{1}{\eta}}, \tag{46}$$

and a continuum of speculative, hyperinflation equilibria indexed by $m_0 \in (0, m^s)$. The time-path for real balances is

$$m_t = \left\{(\bar{m}_t)^{\eta} - e^{\eta(\alpha\sigma\theta+\rho+\pi)t}\left[(\bar{m}_0)^{\eta} - (m_0)^{\eta}\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t), \tag{47}$$

where T solves

$$T = \frac{1}{\alpha\sigma\theta + \rho + \pi} \ln\left[1 - \left(\frac{m_0}{m^s}\right)^{\eta}\right]^{\frac{-1}{\eta}}. \tag{48}$$

An increase in sellers' bargaining power is mathematically identical to an increase in the search frictions. From (48), money loses its value in finite time irrespective of seller's bargaining power. This result, which

[23] A rigorous treatment of the gradual Nash solution and its strategic foundations are presented in Rocheteau et al. (2021).

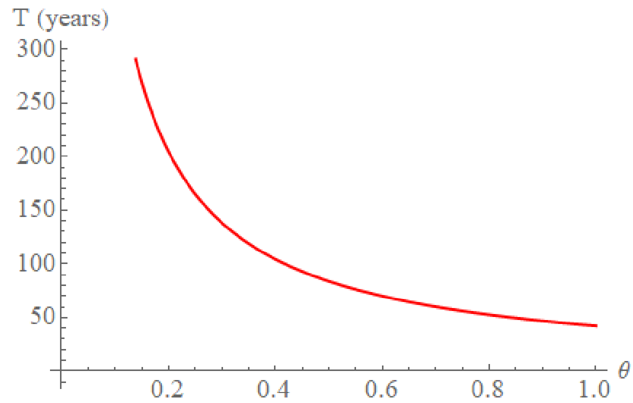


Figure 10: Time for fiat money to become valueless as a function of buyers' bargaining power. Parameter values: $\rho = 0.04$, $\alpha\sigma = 10$, $\eta = 0.02$, $\pi = 0.04$, and $m_0/m^s = 0.99$.

differs from the one obtained under proportional bargaining, is explained by the observation that the marginal value of liquidity,

$$L(y) \equiv \frac{u'(y)}{p'(y)} - 1 = \theta \left[u'(y) - 1 \right],$$

grows unbounded as y tends to zero. From (47), the rate at which the economy diverges from the monetary steady state is $(\alpha\sigma\theta + \rho + \pi)\eta$. So, a higher seller's bargaining power reduces the rate of divergence from the nonspeculative equilibrium.

Figure 10 plots T as a function of θ for parameter values consistent with the calibration strategy used earlier. I set $\alpha\sigma = 10$ so that the frequency of liquidity needs is approximately monthly and $\theta = 0.5$ so that the negotiation is symmetric. I target $|\partial \ln m^s / \partial i| = 10$ which implies $\eta = 0.02$. I set m_0 to 99 percent of the steady-state value. When the negotiation is symmetric, $\theta = 0.5$, it takes about 80 years for fiat money to become valueless. For the same parameter values, if sellers have no bargaining power, $\theta = 1$, then T is divided by two. In contrast, if $\theta = 0.1$, then T is almost 400 years.

5 A fiscal approach to speculative hyperinflations

So far, I assumed that fiscal policy was accommodating monetary policy by adjusting lump-sum transfers or taxes in order to implement an exogenous money growth rate, π_t. There is a large literature arguing that for most hyperinflation episodes, monetary policy is governed by the fiscal needs of the government.[24] As a result, the money growth rate is endogenous and reacts to changes in the value of money in order to achieve a desired seigniorage revenue. I now formalize this view.

[24] According to Sargent and Wallace (1973),

> ‹‹(...) to explain the hyperinflations it is not adequate to regard money creation as exogenous with respect to inflation. Instead, the monetary authorities seemed to make money creation respond directly and systematically to inflation, which was probably an important reason that the hyperinflations developed.››

Following Sargent and Wallace (1973), suppose that the government must finance a flow, $g_t > 0$, of numéraire good with money creation. The dynamics of the money supply is given by

$$\dot{M}_t \phi_t = g_t \Leftrightarrow \pi_t m_t = g_t.$$

The revenue from money creation is equal to the change in the money supply, $\dot{M}_t$, multiplied by the value of money, ϕ_t, or, equivalently, to the money growth rate, π_t, times aggregate real balances, m_t. Substitute $\pi_t = g/m_t$ into the equilibrium ODE, (21),

$$\frac{\dot{m}_t}{m_t} = \rho + \frac{g_t}{m_t} - \alpha\sigma L(m_t). \tag{49}$$

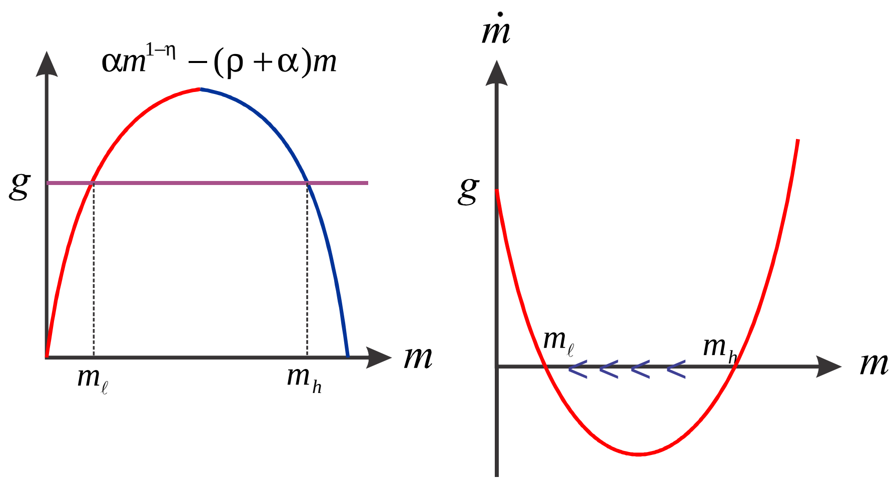


Figure 11: Government purchases, g, financed with money creation. Utility is $u(y) = y^{1-\eta}/(1-\eta)$ with $\eta \in (0,1)$. Left panel: Determination of steady rate. Right panel: Phase diagram with speculative hyperinflations.

The case of CRRA preferences I consider first the generalized CRRA preferences given by (35). As we saw in Section 4.1, this formulation allows us to consider speculative hyperinflations when the coefficient of RRA is greater than one by taking the limit as b tends to zero. For simplicity, $p(y) = y$. In this case, the law of motion for real balances are given by

$$\dot{m}_t = (\rho + \alpha\sigma)\, m_t + g_t - \alpha\sigma m_t (m_t + b)^{-\eta}. \tag{50}$$

Suppose first that $b = 0$, $\eta < 1$, and g_t is proportional to the aggregate production of good y expressed in the numéraire, i.e., $g_t = \gamma\alpha\sigma m_t$. The path for aggregate real balances is obtained by applying Proposition 2 where $\pi_t = \gamma\alpha\sigma$. In a speculative hyperinflation, an increase in γ raises the rate at which real balances diverge from their steady-state value.

In the next proposition, I assume g_t is constant and I distinguish two cases, $\eta < 1$ and $\eta = 1$.

Proposition 6 ***(Dominant fiscal policy under CRRA preferences.)*** *Suppose preferences are given by (35). The government finances a constant flow of consumption, $g > 0$, with money creation.*

1. Low risk aversion. *If $b = 0$, $\eta < 1$, and*

$$g < \bar{g} \equiv \left[\frac{\alpha\sigma(1-\eta)}{\rho+\alpha\sigma}\right]^{\frac{1}{\eta}} \frac{\eta(\rho+\alpha\sigma)}{(1-\eta)}, \tag{51}$$

then there are two steady-state monetary equilibria, $0 < m_\ell^s < m_h^s$, that satisfy $\partial m_\ell^s/\partial g > 0$ and $\partial m_h^s/\partial g < 0$. For all $m_0 \in (m_\ell^s, m_h^s)$, there is an equilibrium such that $m_t \to m_\ell^s$ as $t \to +\infty$. Moreover,

$$m_t = m_h^s - (m_h^s - m_0)e^{\left[\rho+\alpha\sigma-(1-\eta)\alpha\sigma(m_h^s)^{-\eta}\right]t}, \quad \text{if } m_0 \approx m_h^s \text{ and } t \approx 0, \tag{52}$$

$$m_t = m_\ell^s + (m_0 - m_\ell^s)e^{-\left[(1-\eta)\alpha\sigma(m_\ell^s)^{-\eta}-(\rho+\alpha\sigma)\right]t}, \quad \text{if } m_0 \approx m_\ell^s. \tag{53}$$

2. Logarithmic preferences. *Suppose $u(y) = \ln(y + b_n) - \ln(b_n)$ where $\{b_n\}_{n=0}^{+\infty} \in (0,1)^{\mathbb{N}_0}$ is a decreasing sequence such that $b_n \to 0$. Assume $g < \alpha\sigma$. There exists a $N \geq 0$ such that for all $n \geq N$, there are two monetary steady states, $0 < m_{\ell,n}^s < m_{h,n}^s$. As $n \to +\infty$, $m_{\ell,n}^s \to 0$ and*

$$m_{h,n}^s \to m^s \equiv \frac{\alpha\sigma - g}{\rho+\alpha\sigma} > 0. \tag{54}$$

For all $m_0 \in (0, m^s)$, the solution to (49) converges pointwise to

$$\tilde{m}_t = \left[(m_0 - m^s)\,e^{(\rho+\alpha\sigma)t} + m^s\right]\mathbb{I}_{[0,T]}(t), \tag{55}$$

where

$$T = \frac{1}{\rho+\alpha\sigma}\ln\left(\frac{m^s}{m^s - m_0}\right) < +\infty. \tag{56}$$

When η is less than one, the revenue from money creation corresponds to a Laffer curve. If g is lower than some threshold $\bar{g}$ defined in (51), then there are two steady-state monetary equilibria corresponding to two distinct inflation rates. If g is larger than $\bar{g}$, there is no monetary equilibrium with enough seigniorage revenue to finance g.

Speculative hyperinflations correspond to time-paths where the economy converges to the low steady state with low aggregate real balances and high inflation rate. The convergence is only asymptotic as the marginal utility of consumption is finite at the low steady state. From (53), the speed of convergence to the low steady state decreases with m_ℓ^s, and hence decreases with g. Similarly, from (52), the speed of divergence from the high steady state increases with m_h^s, which itself decreases with g.

In the left panel of Figure 12, I illustrate these findings by plotting the time-path of a speculative hyperinflation where $g/m_h^s = 0.04$, i.e., the inflation rate at the high steady state is 4%. The parameter values are the same as in the previous calibrated examples, i.e., $\alpha\sigma = 1$, $\eta = 0.093$, and $m_0/m_h^s = 99\%$. I compare this time-path (represented with a plain blue line) with the one of an economy under a constant money growth rate, $\pi = 0.04$ (represented by a red dashed line). Aggregate real balances are higher for all $t > 0$ under the regime with a dominant fiscal policy. At $T = 70$ years, money has lost all its value under a

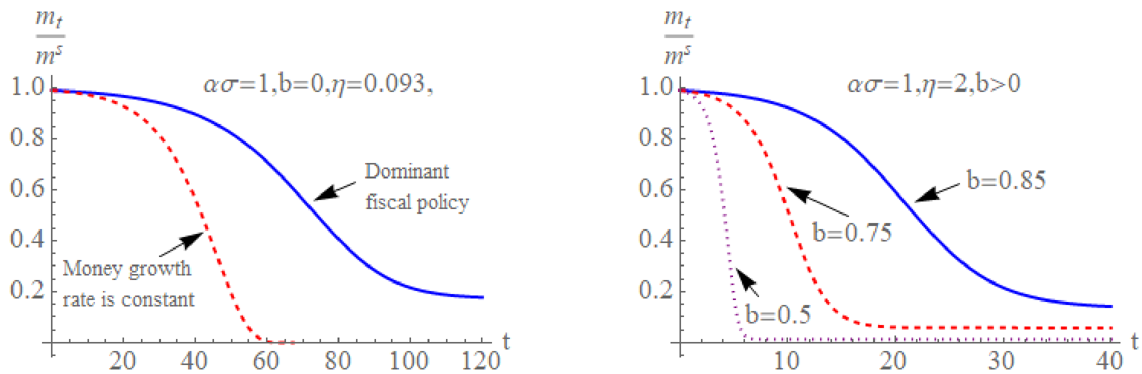


Figure 12: Left: Comparison of speculative hyperinflations under active (blue curve) and passive (red dashed curve) fiscal policy. Right: speculative hyperinflations under general CRRA preferences and active fiscal policy.

passive fiscal policy that implements a constant money growth rate whereas it has kept more than 50% of its steady-state value under an active fiscal policy.

The second part of Proposition 6 analyzes the case where preferences are logarithmic. If $b > 0$ is small, then there are two steady states, m_ℓ^s and m_h^s. As b tends to 0, the low steady state converges to 0. Aggregate real money balances under a speculative hyperinflation converge to a time-path, $\tilde{m}_t$, that reaches 0 in some finite time T. For given m_0, T decreases as g increases whereas for given m_0/m^s, T is independent of g.

While I cannot solve analytically the case with $g > 0$ and $\eta > 1$, it is qualitatively similar to the logarithmic case. For b positive and small, there are two steady-state monetary equilibria. As b goes to 0, speculative hyperinflation equilibria converge to a time-path where money loses its value in finite time. I illustrate this result in the right panel of Figure 12 with an example where $\eta = 2$ and $b \in \{0.5, 0.75, 0.85\}$.

The case of quadratic preferences Finally, there is a functional form for $u(y)$ allowing for closed-form solutions to speculative hyperinflations under a constant seigniorage rule. Indeed, under quadratic preferences, the ODE (49) becomes a Ricatti equation that can be solved analytically.

Proposition 7 ***(Dominant fiscal policy under quadratic preferences)****. Suppose $u(y) = Ay - \varepsilon y^2/2$. If $\alpha\sigma(A-1) - \rho > 0$ and*

$$g < \bar{g} \equiv \frac{[\alpha\sigma(A-1)-\rho]^2}{4\alpha\sigma\varepsilon}, \tag{57}$$

then there are two steady-states monetary equilibria, $m_\ell^s < m_h^s$, and a continuum of speculative hyperinflation equilibria indexed by $m_0 \in (m_\ell^s, m_h^s)$ such that

$$m_t = m_\ell^s + \frac{m_h^s - m_\ell^s}{1 + \left(\frac{m_h^s - m_0}{m_0 - m_\ell^s}\right) e^{\left\{\alpha\sigma(A-1)-\rho-2\alpha\sigma\varepsilon m_\ell^s\right\}t}} \text{ for all } t \geq 0. \tag{58}$$

The speculative term in the denominator of (58) measures how far m_0 is to m_h^s relative to how close it is to m_ℓ^s. The argument of the exponential term decreases with m_ℓ^s, which itself increases with g. So, a higher

g slows down the pace of the hyperinflation. These results are consistent with those obtained under CRRA preferences.

6 Time to dollarize

So far I considered a pure currency economy where fiat money is the only means of payment. In reality, individuals switch to alternative forms of payment (e.g., foreign currencies, precious metals...) when the inflation rate is high. In order to capture this observation, suppose there is an additional asset, called dollar, that can be used as means of payment in a fraction of the transactions. Think of the fiat money as the domestic currency of a small, non-US economy. The real rate of return of the dollar, $r_a \in (-\pi, \rho)$, is constant through time. The government imposes the exclusive use of its currency in a fraction of the transactions. The share of meetings where only domestic fiat money is accepted is equal to $\chi_m \in [0,1]$ while the share of meetings where both money and dollars are accepted is $\chi_2 \equiv 1 - \chi_m$.[25] The pricing mechanism is such that $p(y) = y$.

The value function of a buyer solves

$$\rho V_t^b = \max_{m,a \geq 0} \left\{ -(\rho - r_t)m - (\rho - r_a)a + \alpha\sigma \left[\chi_m \upsilon(m) + \chi_2 \upsilon(m+a) \right] + \zeta_t + \dot{V}_t^b \right\}. \tag{59}$$

The buyer chooses a portfolio of currencies composed of domestic money, m, and dollars, a, both expressed in the numéraire. The opportunity cost of holding dollars is equal to $\rho - r_a$. Dollars serve as means of payment in the fraction χ_2 of matches where both assets are acceptable. The indirect utility in those matches is $\upsilon(m+a)$. The first-order conditions, assuming the choice for m is interior, are

$$-(\rho - r_t) + \alpha\sigma \left[\chi_m \upsilon'(m_t) + \chi_2 \upsilon'(m_t + a_t) \right] = 0 \tag{60}$$

$$-(\rho - r_a) + \alpha\sigma\chi_2 \upsilon'(m_t + a_t) \leq 0 \quad \text{“}=\text{” if } a_t > 0. \tag{61}$$

From (60)-(61), a necessary condition for $a_t > 0$ is $r_a > r_t$. Substituting $\upsilon'(m_t + a_t)$ by its expression given by (61) into (60), using that $r_t = \dot{m}_t / m_t - \pi$, and assuming CRRA preferences, the equilibrium condition is

$$\frac{\dot{m}_t}{m_t} = (\alpha\sigma\chi_m + \rho + \pi) - \alpha\sigma\chi_m (m_t)^{-\eta} - \min\left\{ \alpha\sigma\chi_2 \left[(m_t)^{-\eta} - 1 \right], \rho - r_a \right\}. \tag{62}$$

From the last term on the right side, if the domestic currency is the only means of payment, i.e., $a_t = 0$, then its liquidity value in type-2 meetings cannot be greater than the holding cost of dollars. An equilibrium is a time-path, (m_t, a_t), where m_t solves (62) and, given m_t, a_t solves (61).

Proposition 8 ***(Time to dollarize.)*** *The steady-state monetary equilibrium is such that* $m^s = \min\{m_0^s, m_1^s\}$

[25] For a similar formalization of a dual currency economy, see, e.g., Zhang (2014). For earlier formalizations of the role of the government in the acceptability of fiat money, see, e.g., Aiyagari and Wallace (1997) and Li and Wright (1998).

and $a^s + m^s = \max\{m_0^s, \omega_1^s\}$ *where*

$$m_0^s \equiv \left(\frac{\alpha\sigma}{\alpha\sigma + \rho + \pi}\right)^{\frac{1}{\eta}} \tag{63}$$

$$m_1^s \equiv \left(\frac{\alpha\sigma\chi_m}{\alpha\sigma\chi_m + r_a + \pi}\right)^{\frac{1}{\eta}} \tag{64}$$

$$\omega_1^s \equiv \left(\frac{\alpha\sigma\chi_2}{\rho - r_a + \alpha\sigma\chi_2}\right)^{\frac{1}{\eta}}. \tag{65}$$

There exists a continuum of speculative hyperflation equilibria indexed by $m_0 \in (0, m^s)$. *Along a speculative hyperinflation equilibrium, the time at which the economy starts dollarizing,* $T_0 \equiv \inf\{t \in \mathbb{R}_+ : a_t > 0\}$, *is*

$$\begin{aligned} T_0 &= \frac{1}{\eta(\alpha\sigma + \rho + \pi)} \ln\left[\frac{(m_0^s)^\eta - (\omega_1^s)^\eta}{(m_0^s)^\eta - (m_0)^\eta}\right] \text{ if } \omega_1^s < m_0 < m_0^s \\ &= 0 \text{ otherwise.} \end{aligned} \tag{66}$$

The time at which the economy is fully dollarized is $T_0 + T_1 \equiv \sup\{t \in \mathbb{R}_+ : m_t > 0\}$ *where*

$$T_1 = \frac{\ln\left[1 - (m_{T_0}/m_1^s)^\eta\right]^{-1}}{(\alpha\sigma\chi_m + r_a + \pi)\,\eta}. \tag{67}$$

The time-path for real balances is

$$\begin{aligned} m_t = &\left\{(m_0^s)^\eta - e^{\eta(\alpha\sigma+\rho+\pi)t}\left[(m_0^s)^\eta - (m_0)^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T_0]}(t) \\ &+ \left\{(m_1^s)^\eta - e^{\eta(\alpha\sigma\chi_m + r_a + \pi)(t-T_0)}\left[(m_1^s)^\eta - (m_{T_0})^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[T_0, T_0+T_1]}(t). \end{aligned}$$

The time-path for real holdings of dollars is

$$a_t = (\omega_1^s - m_t)\,\mathbb{I}_{[T_0,+\infty)}(t). \tag{68}$$

In any steady-state equilibrium, real money balances are positive and given by either (63) or (64). Holdings of dollars are positive if the economy features partial dollarization, which occurs if and only if $r_a > \chi_m\rho - \chi_2\pi$, i.e., the rate of return of dollars is greater than a weighted average of the rate of return of an illiquid bond, ρ, and the rate of return of the domestic currency, $-\pi$, where the weights are given by the acceptability parameters, χ_m and χ_2. From (64) and (65), a measure of dollarization is

$$\text{dollarization} \equiv \frac{a^s}{a^s + m^s} = 1 - \left(\frac{\chi_m(\rho - r_a) + \alpha\sigma\chi_2\chi_m}{\chi_2(r_a + \pi) + \alpha\sigma\chi_2\chi_m}\right)^{\frac{1}{\eta}}. \tag{69}$$

It increases with r_a, π, and χ_2. If r_a is not large enough relative to $-\pi$, then there is no dollarization at the nonspeculative, steady-state monetary equilibrium and, from (63), the steady-state real balances are the same as the ones in a pure currency economy.

A speculative hyperinflation has two stages.[26] From $t = 0$ to T_0, the real holdings of the domestic currency shrink gradually but agents do not accumulate dollars, i.e., the economy remains undollarized. This first

[26] I treat the case $\alpha\sigma\chi_m + r_a + \pi = 0$ separately in the proof of the proposition.

stage resembles the speculative hyperinflation of a pure currency economy. From (66), the duration of the first stage, T_0, shrinks as r_a increases. The second stage starts at time T_0 where the economy begins to dollarize. From T_0 to $T_0 + T_1$ agents substitute their real units of domestic currency for dollars while maintaining their liquid wealth at ω_1^s. At time $T_0 + T_1$, the real holdings of domestic currency reach 0 at which point the economy is fully dollarized, i.e., $a_{T_0+T_1} > 0$ and $m_{T_0+T_1} = 0$. So T_1 is the duration of the dollarization process. The speed at which real balances depart from m_1^s in the second phase is given by $\eta(\alpha\sigma\chi_m + r_a + \pi)$. It increases with χ_m, the share of transactions that can only be executed with the domestic currency, and the rate of return of dollars, r_a, and the money growth rate, π. So, economies with a high $r_a + \pi$ are highly dollarized at the steady state and, if they are subject to a speculative hyperinflation, they complete the dollarization at a high pace. On the contrary, economies with high χ_m have a low level of dollarization at the nonspeculative steady state, but they dollarize at a high speed along a speculative hyperinflation equilibrium. For a given (m_{T_0}/m_1^s), the time to dollarize the economy shrinks as r_a or π increases.

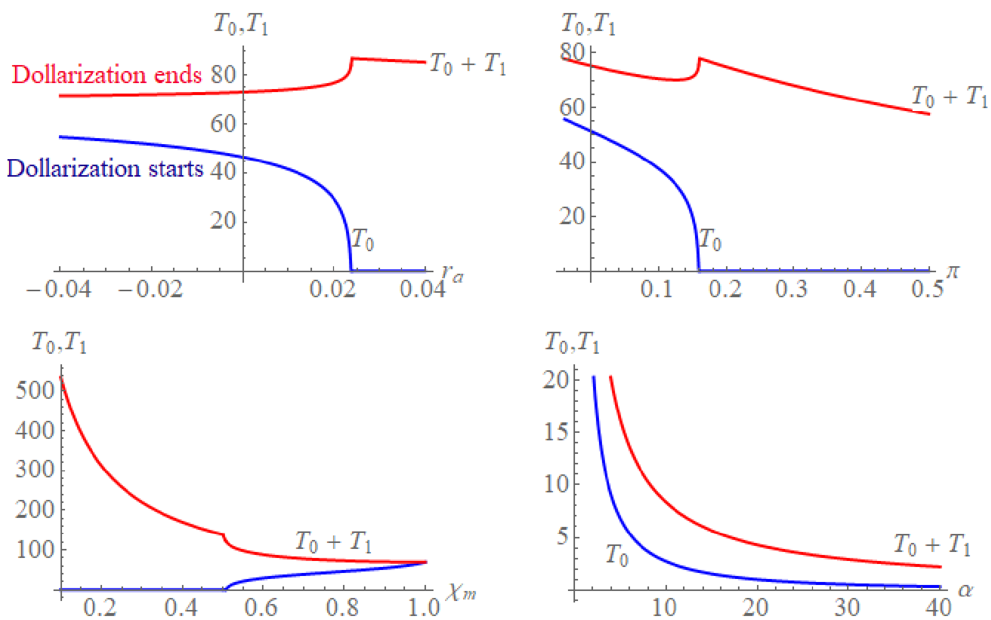


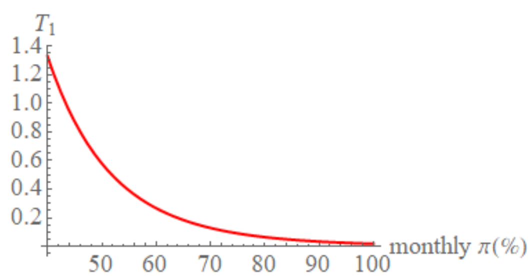
Figure 13: Time to dollarize. Dollarization starts at T_0 and ends at $T_0 + T_1$. Parameter values: $\alpha\sigma = 1$, $\rho = 0.04$, $\eta = 0.093$, $\chi_m = 0.8$, $\pi = 0.04$, $r_a = 0$. The ratio of m_0/m^s is set to 0.99.

In Figure 13, I plot the time to dollarize during a speculative hyperinflation and I show how it varies with policy parameters and fundamentals. The parameter values are the same as in Section 3.4 with $\alpha\sigma = 1$, $\eta = 0.093$, and $\pi = 0.4$. In terms of the new parameters, I set $r_a = 0$, i.e., the competing asset has a rate of return equal to zero (if it is a currency, it features zero inflation), and $\chi_m = 0.8$, i.e., the competing assets is accepted in 20% of the meetings. I keep the ratio m_0/m^s constant and equal to 0.99 across these experiments.

If $\chi_m = 1$, the model is identical to the one in Section 3.4 and $T_0 + T_1 = 70$ years. It can be seen from the bottom left panel that the competition with another asset raises the time it takes for the domestic currency to fully disappear. If χ_m falls to 50 percent, then $T_0 + T_1$ doubles to almost 140 years. If $\chi_m = 10\%$, i.e., the competition takes place in most meetings, then $T_0 + T_1$ is greater than 500 years. As χ_m decreases, T_0 decreases and the dollarization of the economy starts sooner. However, the duration of the dollarization stage, T_1, gets longer.

In the top left panel, the length of time during which the economy remains undollarized, T_0, decreases with the rate of return of the competing asset, r_a. However, the duration of the dollarization stage, T_1, and the overall time it takes for the domestic currency to die, $T_0 + T_1$, increases with r_a as long as $T_0 > 0$. When r_a is sufficiently large so that $T_0 = 0$, then T_1 starts decreasing with r_a. This result is explained by the fact that at $t = 0$, the economy is already partially dollarized and the extent of the dollarization at the steady state increases with r_a.

The top right panel describes how T_0 and T_1 depend on the money growth rate, π. The results are qualitatively similar to the ones obtained for r_a. As the rate of return of the domestic currency deteriorates, i.e., π increases, the economy remains undollarized for a shorter period of time. While the duration of the dollarization stage, T_1, increases with π when the economy is undollarized initially, it decreases with π when the economy is dollarized at the steady state. Finally, as shown in the bottom right panel, if the frequency of meetings increases, dollarization starts earlier and the domestic currency dies sooner.



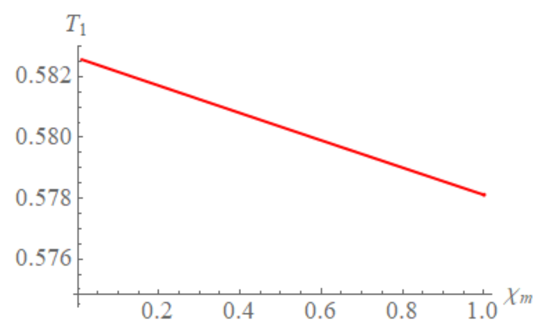


Figure 14: Time to dollarize. Parameter values: $\alpha\sigma = 1$, $\rho = 0.04$, $\eta = 0.093$, $\chi_m = 0.8$, $\pi = (1{+}50/100)^{12}{-}1$ (i.e., monthly money growth rate is 50 percent), $r_a = 0$. The ratio of m_0/m^s is set to 0.99.

In Figure 14, I report the time for the economy to fully dollarize when the monthly money growth is in the hyperinflation range, i.e., 50 percent and above. As shown in the right panel, the degree of competition between the domestic currency and the competing asset only has a marginal effect on T_1. The reason is that at such levels of money growth, the economy is almost entirely dollarized at the steady state. It takes less than a year for the residual amount of domestic real balances to fully disappear.

7 When credit dies

So far I studied speculative hyperinflation in economies with fiat monies. As the use of fiat money in developed economies recedes, do speculative hyperinflations become less relevant? I now describe a phenomenon similar to speculative hyperinflations in a pure credit economy with limited commitment, as in Kehoe and Levine (1993).[27]

I maintain the assumption from the pure currency economy that there is no technology to enforce the repayment of debts, i.e., repayment has to be self-enforcing. In order for credit arrangements to be incentive feasible, there is a record-keeping technology that keeps track of transactions and repayments. In equilibrium, buyers will have incentives to repay their debts in order to maintain their access to credit. I assume that the record-keeping technology is imperfect in the following sense. A default event is recorded with probability $\lambda \in [0,1]$. Hence, there is a probability $1-\lambda$ that an agent who defaulted is not reported publicly and hence is not excluded from future credit.[28]

Credit transactions take place as follows. At times T_n, $n \in \mathbb{N}$, a buyer receives opportunities to consume by being matched bilaterally with sellers. While she cannot produce in the match, she can promise to repay her debt as soon as the meeting is over, at time T_n^+. The maximum amount a buyer can promise to repay at time t expressed in the numéraire, called *debt limit*, is denoted d_t. This real payment capacity, which is analogous to m_t in the monetary economy, is endogenous.

The expected lifetime utility of a buyer solves the following HJB equation:

$$\rho V_t^b = \alpha\sigma\upsilon(d_t) + \dot{V}_t^b. \tag{70}$$

The interpretation is similar to (18) where m_t is replaced with d_t. The debt limit is defined as the highest amount of debt that a buyer would repay willingly knowing that if she defaults she will be excluded from future transactions with probability λ, in which case her lifetime utility is 0 (under the assumption that $u(0) = 0$). It solves

$$d_t = \lambda V_t^b. \tag{71}$$

By defaulting, the buyer saves the cost of repayment, d_t, but incurs the cost of being excluded from future credit, V_t^b, with probability λ. Substitute d_t by its expression given by (71) into (70) to obtain

$$\rho d_t = \alpha\sigma\lambda\upsilon(d_t) + \dot{d}_t. \tag{72}$$

An equilibrium is a time-path, d_t, solution to (72).

In order to solve for equilibria in closed form, I make the following assumptions. Buyers make take-it-or-leave-it offers to sellers, i.e., $p(y) = y$, and preferences are of the type $u(y) = y^{1-\eta}$ with $\eta \in (0,1)$. In order to keep the analysis succinct, I focus on the region in the parameter space where the debt limit at the

[27] Such equilibria have been described in a discrete-time competitive economy by Bloise, Reichlin, and Tirelli (2013).

[28] A characterization of the perfect Bayesian equilibria of such a pure credit economy is provided in Bethune, Hu, and Rocheteau (2018).

steady state, d^s, is less than y^*, which requires $\rho y^* > \alpha\sigma\lambda\,[u(y^*) - y^*]$. This condition holds is agents are sufficiently impatient. The ODE (72) can be rewritten as:

$$\dot{d}_t = (\rho + \alpha\sigma\lambda)\, d_t - \alpha\sigma\lambda(d_t)^{1-\eta}. \tag{73}$$

This ODE is formally identical to the ODE in the STW model, (6), where $1 - M$ has been replaced with λ. By the same logic as in Proposition 1, I obtain the following proposition.

Proposition 9 ***(Speculative debt limits.)*** *Consider a pure credit economy under limited commitment. The positive steady state is*

$$d^s = \left(\frac{\alpha\sigma\lambda}{\rho + \alpha\sigma\lambda}\right)^{\frac{1}{\eta}}. \tag{74}$$

In addition, there are a continuum of nonstationary equilibria, indexed by $d_0 \in (0, d^s)$*, such that*

$$d_t = \left\{(d^s)^\eta - [(d^s)^\eta - (d_0)^\eta]\, e^{\eta(\rho+\alpha\sigma\lambda)t}\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T]}(t) \tag{75}$$

where

$$T = \frac{-\ln\left[1 - (d_0/d^s)^\eta\right]}{\eta\,(\rho + \alpha\sigma\lambda)}. \tag{76}$$

Along those speculative equilibria, the real borrowing capacity of buyers goes to zero in finite time. The rate at which d_t diverges from its steady-state value, $\eta\,(\rho + \alpha\sigma\lambda)$, increases with the reliability of the record-keeping technology.

8 Conclusion

I showed that a textbook model of a monetary economy in continuous time predicts that a fiat money experiencing a speculative hyperinflation dies in finite time. The duration of fiat money depends on the initial value agents coordinate on, fundamentals, such as the trading technology and preferences, and policies, such as the money growth rate and the size of the government budget deficit financed with money creation. For parameter values consistent with the US money demand, money dies in 90 years if its initial value departs from its steady state by a mere one percent and the annual money growth rate is constant and equal to 4 percent. If the monthly money growth is larger than 50 percent – which corresponds to hyperinflation territory – then the life expectancy of fiat money drops to about 7 years. In the case of multiple, competing means of payment, a speculative hyperinflation generates a full dollarization of the economy in finite time. However, the currency competition allows the domestic currency to last longer. All these results shed some light on the role of market structure, market power, and policies for the dynamics of hyperinflation. These insights are not specific to pure currency economies and also apply to credit economies when agents have limited commitment.

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Appendix A: Proofs of propositions

Proof of Proposition 2. I solve the ODE (22) over a time interval, $[0, T)$, such that $m_t > 0$ for all $t \in [0, T)$. I operate the change of variable, $x_t = m_t^\eta$, and I consider solutions such that $m_t < y^* = 1$ for all t, i.e., $x_t \in [0, 1)$ for all t. By differentiating x_t with respect to t, I obtain $\dot{x}_t = \eta m_t^{\eta-1} \dot{m}_t$. Substitute $\dot{m}_t = \dot{x}_t m_t^{1-\eta}/\eta$ into (22) and rearrange to obtain the following nonautonomous, linear differential equation:

$$\dot{x}_t = (\alpha\sigma + \rho + \pi_t)\eta x_t - \alpha\sigma\eta.$$

I solve this ODE using the method of the integrating factor. Multiply both sides of the ODE by $e^{-\eta(\alpha\sigma+\rho)t - \eta\Pi(t)}$, where $\Pi(t) = \int_0^t \pi_s ds$ is the cumulative inflation rate, and integrate forward from t to T:

$$x_t = \int_t^T e^{-\eta[(\alpha\sigma+\rho)(s-t)+\Pi(s)-\Pi(t)]}\alpha\sigma\eta ds + e^{-\eta[(\alpha\sigma+\rho)(T-t)+\Pi(T)-\Pi(t)]}x_T. \tag{77}$$

Consider first the "nonspeculative" solution such that $x_t > 0$ for all t. I take the limit as T goes to $+\infty$. From the restriction to time-paths such that $x_t \leq 1$ and the assumption $\pi_t + \rho > 0$ for all t, the limit of the second term on the right side of (77) is

$$\lim_{T\to+\infty} e^{-\eta[(\alpha\sigma+\rho)(T-t)+\Pi(T)-\Pi(t)]}x_T = 0.$$

Hence, from (77),

$$\bar{x}_t \equiv \lim_{T\to+\infty} x_t = \int_t^{+\infty} e^{-\eta[(\alpha\sigma+\rho)(s-t)+\Pi(s)-\Pi(t)]}\alpha\sigma\eta ds.$$

Since $\Pi(s) - \Pi(t) \geq -\rho(s-t)$, the term between squared brackets in the exponential function underneath the integral is positive for all $s > t$ and hence $\bar{x}_t$ is bounded above by

$$\int_t^{+\infty} e^{-\eta\alpha\sigma(s-t)}\alpha\sigma\eta ds = 1.$$

Moreover, $\Pi(s) - \Pi(t) \leq (s-t)\bar{\pi}$ so that $\bar{x}_t$ is bounded below by

$$\int_t^{+\infty} e^{-\eta(\alpha\sigma+\rho+\bar{\pi})(s-t)}\alpha\sigma\eta ds = \frac{\alpha\sigma}{\alpha\sigma+\rho+\bar{\pi}} > 0.$$

So, $\bar{x}_t \in (0, 1)$ for all t and hence it is a solution. Using that $\bar{x}_t = \bar{m}_t^\eta$, I obtain (25).

The speculative hyperinflation equilibria correspond to the continuum of solutions, indexed by $T \in (0, +\infty)$, where money becomes valueless at time $T < +\infty$, $x_T = 0$. From (77),

$$x_t = \alpha\sigma\eta \int_t^T e^{-\eta\{(\alpha\sigma+\rho)(s-t)+[\Pi(s)-\Pi(t)]\}}ds \leq \bar{x}_t \text{ for all } t \leq T.$$

Given $T > 0$, the initial value for x is

$$x_0 = \alpha\sigma\eta \int_0^T e^{-\eta[(\alpha\sigma+\rho)s+\Pi(s)]}ds.$$

Using that $m_t = (x_t)^{\frac{1}{\eta}}$, I obtain (24). At $t = T$, $m_t = 0$ and hence, from (22), $\dot{m}_t = 0$. It follows that $m_t = 0$ for all $t \geq T$. ■

Proof of Proposition 3. From (21), assuming $y_t \leq m_t$ binds for all t, the law of motion for real balances are given by

$$\dot{m}_t = f(m_t), \tag{78}$$

where

$$f(x) \equiv (\rho + \pi + \alpha\sigma)\, x - \alpha\sigma x u'(x).$$

The positive steady state, when $\dot{m}_t = 0$, is given by (31). The left side of (31) is decreasing in m^s, it is equal to $u'(0) < +\infty$ when $m^s = 0$, and it is equal to one when $m^s = y^* < +\infty$. The right side is constant and greater than one. Hence, a solution exists provided that the left side evaluated at $m^s = 0$ is greater than the right side, i.e., $\alpha\sigma u'(0) > \alpha\sigma + \rho + \pi$.

Existence and uniqueness for arbitrary initial conditions. The phase diagram representing the ODE is similar to the one in Figure 6. It can be seen that any solution to $\dot{m}_t = f(m_t)$ with $m_0 \in [0, m^s]$ is such that $m_t \in [0, m^s]$ for all $t \in \mathbb{R}$. Consider an open set $\Omega \subset \mathbb{Y}$ such that $[0, m^s] \subset \Omega$. Since u' is C^∞, the function f is continuously differentiable with

$$f'(x) = \rho + \pi + \alpha\sigma - \alpha\sigma u'(x) - \alpha\sigma x u''(x) \in (-\infty, +\infty) \;\; \forall x \in \Omega.$$

Since $f : \Omega \to \mathbb{R}$ is continuously differentiable, then it is locally Lipschitz continuous for all $x \in \Omega$, i.e., for all $x \in \Omega$, there is a neighborhood $V \subset \Omega$ and a $L > 0$ such that

$$|f(y) - f(z)| \leq L\,|y - z|\,, \text{ for all } y, z \in V.$$

By the Cauchy-Lipschitz theorem, for all initial conditions $m_0 \in [0, m^s]$, $\dot{m}_t = f(m_t)$ has a unique solution, $m_t : \mathbb{R} \to [0, m^s]$. It implies that if $m_0 \in (0, m^s)$, m_t cannot converge to the steady state solution, $m_t \equiv 0$, in finite time.

Approximate solutions. I linearize the ODE in the neighborhood of $m_t = 0$ to obtain:

$$\dot{m}_t = \left[\rho + \pi + \alpha\sigma - \alpha\sigma u'(0)\right] m_t,$$

where I used that $u''(x)$ is differentiable at $x = 0$ so that $u''(0) \in (-\infty, 0)$ and $\lim_{x \to 0} x u''(x) = 0$. The solution is (33). Under the condition for the existence of a positive steady state, i.e., $\alpha\sigma u'(0) > \alpha\sigma + \rho + \pi$, the term on the right side between squared brackets is negative, $\partial \dot{m}_t / \partial m_t \in (-\infty, 0)$. Hence, m_t converges to 0 but only at the limit as $t \to +\infty$.

Suppose $m_0 \approx m^s$ and $t \approx 0$ so that m_t is in the neighborhood of m^s. The linearization of the ODE, (78), in that neighborhood gives

$$\dot{m}_t = \left\{(\rho + \pi + \alpha\sigma) - \alpha\sigma u'(m^s) - \alpha\sigma m^s u''(m^{ss})\right\}(m_t - m^s).$$

From (31), $(\rho + \pi + \alpha\sigma) = \alpha\sigma u'(m^s)$, and hence

$$\dot{m}_t = \alpha\sigma u'(m^s) \frac{-m^s u''(m^{ss})}{u'(m^s)} (m_t - m^s).$$

Using the notation $\eta(m^s) \equiv -m_s u''(m^s)/u'(m^s)$ and $\alpha\sigma u'(m^s) = \rho + \pi + \alpha\sigma$, one obtains

$$\dot{m}_t = (\alpha\sigma + \rho + \pi)\, \eta(m^s)\, (m_t - m^s)\, .$$

The solution to this linear differential equation is (32). ■

Proof of Corollary 4.

Part 1. Under the generalized CRRA preferences, for every element of the sequence $\{b_n\}_{n=0}^{+\infty}$, the ODE, (21), can be rewritten as

$$\dot{m}_{t,n} = (\rho + \pi + \alpha\sigma)\, m_{t,n} - \alpha\sigma m_{t,n}(m_{t,n} + b_n)^{-\eta}. \tag{79}$$

The positive steady state, denoted m_n^s, solves $\dot{m}_{t,n} = 0$ and $m_{t,n} > 0$, i.e., $\alpha\sigma(m_n^s + b_n)^{-\eta} = \rho + \pi + \alpha\sigma$. Solving for m_n^s in closed form gives (36). Using that $\{b_n\}_{n=0}^{+\infty}$ is decreasing, $\{m_n^s\}_{n=0}^{+\infty}$ is increasing.

Part 2. The uniqueness of the solution to the ODE, (79), follows from Proposition 3 and the fact that the right side of (79) is continuously differentiable for all $m_{t,n} > -b_n$. Since $\{m_n^s\}_{n=0}^{+\infty}$ is increasing, the condition $m_0 < m_0^s$ guarantees that $m_{0,n} = m_0 < m_n^s$ for all n. From Proposition 3, for all $m_{0,n} \in (0, m_n^s)$, $\dot{m}_{t,n} < 0$ and $m_{t,n} \to 0$ as $t \to +\infty$. Moving backward in time, $m_{t,n} \to m_n^s$ as $t \to -\infty$.

Part 3. From (79), $\dot{m}_{t,n}$ is a continuously differentiable function of $m_{t,n}$ and b_n for all $(m_{t,n}, b_n) \in \mathbb{R}_+^2$ such that $m_{t,n} + b_n > 0$. By the theorem of continuous dependence (see, e.g., Grant, 2014, page 20), the solution to (79) is continuous in b_n. As $b_n \to 0$, the ODE (79) converges to

$$\dot{m}_t = (\rho + \pi + \alpha\sigma) - \alpha\sigma m_t (m_t)^{-\eta}.$$

Hence, as $b_n \to 0$, $m_{t,n} : [0, T] \to [0, m_0]$ converges pointwise to the solution in Corollary 3. ■

Proof of Proposition 4. Under quadratic preferences, the ODE, (78), becomes

$$\dot{m}_t = [\rho + \pi + \alpha\sigma(1 - A)]\, m_t + \alpha\sigma\varepsilon\, (m_t)^2\, . \tag{80}$$

The positive steady state solves $\dot{m}_t = 0$ and $m_t > 0$, which gives (40). In order for $m^s > 0$, $\alpha\sigma A > \alpha\sigma + \rho + \pi$, which can be rewritten as (39). The ODE, (80), is a Bernoulli equation. Assuming $m_t > 0$, I adopt the change of variable $x = m^{-1}$. Then $\dot{x}_t = -\dot{m}_t/(m_t)^2$. Substitute $\dot{m}_t = -\dot{x}_t (m_t)^2$ into (80) to obtain

$$\dot{x}_t = -\left[\rho + \pi + \alpha\sigma(1 - A)\right] x_t - \alpha\sigma\varepsilon.$$

The solution to this linear ODE is

$$x_t = x^s + (x_0 - x^s)e^{[\alpha\sigma(A-1) - \rho - \pi]t},$$

where $x^s = 1/m^s$. Using that $m_t = 1/x_t$, I obtain (41). It is easy to check that for all $m_0 \in (0, m^s)$, $m_t > 0$ for all $t > 0$, as conjectured above. ■

Proof of Proposition 5. Given the pricing function, (42), and assuming the liquidity constraint binds at all dates, the law of motion for real balances, (21), can be reexpressed as

$$\dot{m}_t = f(m_t). \tag{81}$$

where

$$f(m) \equiv (\rho + \pi + \alpha\sigma)\, m - \alpha\sigma \frac{(m)^{1-\eta}}{\theta + (1-\theta)(m)^{-\eta}}.$$

The positive steady state obtained from $\dot{m}_t = 0$ and $m_t > 0$ solves (43). A solution exists provided that $\alpha\sigma > (\rho + \pi + \alpha\sigma)(1-\theta)$. From the definition of f above, it can be checked that f is continuously differentiable with

$$f'(m) = \rho + \pi + \alpha\sigma - \alpha\sigma \frac{(1-\eta)\theta(m)^{\eta} + (1-\theta)}{\left[\theta(m)^{\eta} + (1-\theta)\right]^2}.$$

In particular,

$$f'(0) = \rho + \pi - \frac{\alpha\sigma\theta}{1-\theta} \in (-\infty, 0)\,.$$

By the logic of Proposition 3, there exist a continuum of speculative, hyperinflation equilibria indexed by $m_0 \in (0, m^s)$ featuring $m_t > 0$ and $\dot{m}_t < 0$ for all $t > 0$ with $m_t \to 0$ as time goes to $+\infty$. In the neighborhood of $m_t = 0$, the ODE (81) can be linearized to obtain

$$\dot{m}_t = \left(\rho + \pi - \frac{\alpha\sigma\theta}{1-\theta}\right) m_t.$$

The solution to this linear ODE corresponds to (44). ■

Proof of Proposition 6. Part 1.

Steady states. From (50), steady-state equilibria solve

$$\alpha\sigma(m^s)^{1-\eta} - (\rho + \alpha\sigma)\, m^s = g. \tag{82}$$

The left side of (82) is strictly concave, equal to 0 when $m^s = 0$ and $m^s = \left[\alpha\sigma/(\rho+\alpha\sigma)\right]^{1/\eta}$. It reaches a maximum when $m^s = m_{\max} \equiv \left[\alpha\sigma(1-\eta)/(\rho+\alpha\sigma)\right]^{1/\eta}$. The right side is constant and equal to g. Hence, if the left side when evaluated at $m_{\max}$ is greater than g, i.e.,

$$g < \left[\frac{\alpha\sigma(1-\eta)}{\rho+\alpha\sigma}\right]^{\frac{1}{\eta}} \frac{\eta(\rho+\alpha\sigma)}{(1-\eta)},$$

then there are two steady-state equilibria, $0 < m^s_\ell < m^s_h$. Because the left side of (82) is increasing in m^s for all $m^s < m_{\max}$, and $m^s_\ell < m_{\max}$, an increase in g raises m^s_ℓ. By a symmetric reasoning, an increase in g reduces m^s_h.

Speculative equilibria. From (50), $\partial\dot{m}_t/\partial m_t < 0$ for all $m_t < m_{\max}$. From the result that $m^s_\ell < m_{\max}$ and $\dot{m}_t = 0$ when $m_t = m^s_\ell$, it follows that $\dot{m}_t > 0$ for all $m_t < m^s_\ell$, and hence m_t converges to m^s_ℓ. So there are no equilibria where the value of money converges to 0. By a similar reasoning, for all $m_0 \in (m^s_\ell, m^s_h)$,

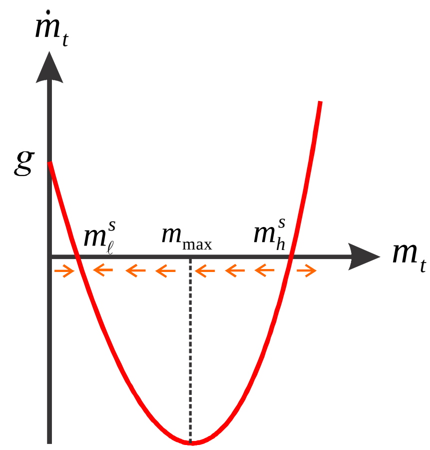

Figure 15: Phase diagram of ODE (50)

$\dot{m}_t < 0$. So there are a continuum of inflationary paths where m_t converges to m^s_ℓ. These dynamics are illustrated in the phase diagram in Figure 15.

<u>*Approximation*</u>. From (50),

$$\left.\frac{\partial \dot{m}_t}{\partial m_t}\right|_{m_t = m^s_\ell} = \rho + \alpha\sigma - (1-\eta)\alpha\sigma(m^s_\ell)^{-\eta}$$

I use this expression to linearize (50) in the neighborhood of m^s_ℓ:

$$\dot{m}_t = \left[\rho + \alpha\sigma - (1-\eta)\alpha\sigma(m^s_\ell)^{-\eta}\right](m_t - m^s_\ell).$$

Using that $\partial \dot{m}_t / \partial m_t < 0$ at $m_t = m^s_\ell$, the solution for m_0 in the neighborhood of m^s_ℓ converges to m^s_ℓ. The closed-form solution to the linear ODE above is (53). Similarly, I linearize (50) in the neighborhood of m^s_h to obtain (52).

<u>Part 2</u>. If $u(y) = \ln(y+b) - \ln(b)$, with $b \in (0,1)$, then the ODE for m_t, (49), is rewritten as

$$\dot{m}_t = (\rho + \alpha\sigma)\, m_t + g - \frac{\alpha\sigma m_t}{m_t + b}. \tag{83}$$

A steady state solves $\dot{m}_t = 0$, i.e.,

$$(\rho + \alpha\sigma)\, m^s + g = \frac{\alpha\sigma m^s}{m^s + b}.$$

The left-hand side (LHS) is linear increasing in m^s with a positive intercept. The right-hand side (RHS) is a strictly increasing and strictly concave function of m^s with a slope given by

$$\frac{\partial RHS}{\partial m^s} = \frac{\alpha\sigma b}{(m^s + b)^2}.$$

So, $\partial RHS/\partial m^s|_{m^s=0} = \alpha\sigma/b$ and $\partial RHS/\partial m^s|_{m^s \to +\infty} = 0$. Moreover,

$$\frac{\partial RHS}{\partial m^s} = \frac{\partial LHS}{\partial m^s} \iff \frac{\alpha\sigma b}{(m^s + b)^2} = \rho + \alpha\sigma.$$

The solution is $m^s = \hat{m}$ where

$$\hat{m} \equiv \sqrt{\frac{\alpha\sigma b}{\rho + \alpha\sigma}} - b.$$

Hence, $\hat{m} \in (0, 1-b)$ if $b < \alpha\sigma/(\rho + \alpha\sigma)$. If this condition does not hold, $\partial RHS/\partial m^s < \partial LHS/\partial m^s$ for all m^s. Finally,

$$RHS|_{m^s=0} = 0 < LHS|_{m^s=0} = g$$

and

$$RHS|_{m^s=1-b} = \alpha\sigma(1-b) < LHS|_{m^s=1-b} = \alpha\sigma(1-b) + \rho(1-b) + g.$$

Hence, if a solution exists, then there are two solutions, $m^s_\ell \in (0, \hat{m}]$ and $m^s_h \in [\hat{m}, 1-b)$. The determination of the steady states are represented graphically in Figure 16.

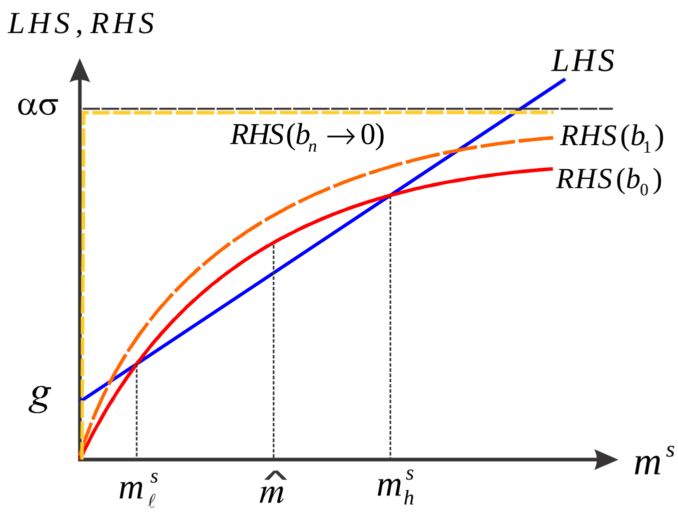

Figure 16: Determination of steady states under logarithmic preferences

Consider a decreasing sequence, $\{b_n\}_{n=0}^{+\infty}$, that converges to 0. As b_n approaches zero, the right side increases and approaches $\alpha\sigma$ for all $m^s > 0$, as shown in Figure 16. Hence, if $g < \alpha\sigma$, there is a $N \geq 0$ such that for all $n \geq N$, there are two steady states, $m^s_{\ell,n}$ and $m^s_{h,n}$. Since RHS is decreasing in b, $m^s_{\ell,n}$ is decreasing in n and $m^s_{h,n}$ is increasing in n. Moreover, by the squeeze theorem, since $0 \leq m^s_{\ell,n} \leq \hat{m}(b_n)$ and $\lim_{b\to 0}\hat{m} = 0$, it follows that $\lim_{n\to+\infty} m^s_{\ell n} = 0$. The high steady state converges to the solution to $(\rho + \alpha\sigma)\, m^s + g = \alpha\sigma$, i.e., (54).

Using that $m^s_{\ell n} \searrow 0$ and $m^s_{hn} \nearrow m^s$, for all $m_0 \in (0, m^s)$, there is a $\tilde{N} \geq 0$ such that for all $n \geq \tilde{N}$, $m_0 \in (m^s_{\ell,n}, m^s_{h,n})$. The ODE, (83), can be written as $\dot{m}_t = f(m_t; b)$ where

$$f(m; b) \equiv (\rho + \alpha\sigma)\, m + g - \frac{\alpha\sigma m}{m + b}, \text{ for all } m \geq 0.$$

Since $f \in C^1$ for all $m > -b$, it is locally Lipschitz continuous. By the theorem of Cauchy-Lipschitz, the ODE (83) admits a unique solution, $m_{t,n}$, given the initial condition, m_0, and it is such that $m_{t,n}$ converges to $m^s_{\ell,n}$ as $t \to +\infty$. Since f is continuously differentiable with respect to m and b for all (m, b) such that $m + b > 0$, by the theorem of continuous dependence (see, e.g., Grant, 2014, page 20), the solution to the ODE is continuous in b_n. As $b_n \to 0$, the ODE converges to

$$\dot{m}_t = (\rho + \alpha\sigma)\, m_t + g - \alpha\sigma, \text{ for all } m_t > 0.$$

The solution to this linear ODE is $\tilde{m}_t$ given by (55) where T is defined by $\tilde{m}_T = 0$. So, for all $t \in (0, T)$, $m_{t,n}$ converges pointwise to $\tilde{m}_t$ as $n \to +\infty$. ■

Proof of Proposition 7. The ODE (49) becomes

$$\dot{m}_t = \alpha\sigma\varepsilon(m_t)^2 - [\alpha\sigma(A-1) - \rho]\, m_t + g. \tag{84}$$

The steady-state solutions to (84) are given by

$$\begin{aligned} m^s_h &= \frac{\alpha\sigma(A-1) - \rho + \sqrt{[\alpha\sigma(A-1)-\rho]^2 - 4\alpha\sigma\varepsilon g}}{2\alpha\sigma\varepsilon} \\ m^s_\ell &= \frac{\alpha\sigma(A-1) - \rho}{\alpha\sigma\varepsilon} - m^s_h \end{aligned}$$

Positive solutions exist if $\alpha\sigma(A-1) - \rho > 0$ and

$$g < \frac{[\alpha\sigma(A-1)-\rho]^2}{4\alpha\sigma\varepsilon}.$$

The ODE (84) is a Ricatti equation that admits an explicit solution. Denote $z_t = m_t - m^s_\ell$ to rewrite (84) as

$$\dot{z}_t = \alpha\sigma\varepsilon(z_t)^2 + \{2\alpha\sigma\varepsilon m^s_\ell - [\alpha\sigma(A-1) - \rho]\}\, z_t.$$

The ODE in z_t is a Bernoulli equation. Using the change of variable $x_t = 1/z_t$, it becomes

$$\dot{x}_t = -\alpha\sigma\varepsilon - \{2\alpha\sigma\varepsilon m^s_\ell - [\alpha\sigma(A-1) - \rho]\}\, x_t.$$

Using that $\alpha\sigma\varepsilon m^s_\ell + \alpha\sigma\varepsilon m^s_h = \alpha\sigma(A-1) - \rho$, the solution is

$$x_t = \frac{1}{m^s_h - m^s_\ell} + \left(x_0 - \frac{1}{m^s_h - m^s_\ell}\right) e^{\{\alpha\sigma(A-1) - \rho - 2\alpha\sigma\varepsilon m^s_h\}t}.$$

Using that $x_0 = 1/(m_0 - m^s_\ell)$ and $m_t = 1/x_t + m^s_\ell$, the solution to (84) is (58). ■

Proof of Proposition 8. The ODE (62) can be rewritten as

$$\frac{\dot{m}_t}{m_t} = \max\{\Gamma_0(m_t), \Gamma_1(m_t)\},$$

where

$$\begin{aligned} \Gamma_0(m_t) &\equiv (\alpha\sigma + \rho + \pi) - \alpha\sigma(m_t)^{-\eta} \\ \Gamma_1(m_t) &\equiv (\alpha\sigma\chi_m + r_a + \pi) - \alpha\sigma\chi_m(m_t)^{-\eta}. \end{aligned}$$

From (63), m_0^s is the unique solution to $\Gamma_0(m_0^s)=0$. From (64), m_1^s is the unique solution to $\Gamma_1(m_1^s)=0$. The right side of the ODE is increasing in m_t with

$$\lim_{m_t \searrow 0} \frac{\dot{m}_t}{m_t} = -\infty \text{ and } \left.\frac{\dot{m}_t}{m_t}\right|_{m_t=1} = \rho+\pi > 0.$$

Hence, there exists a unique positive steady state, $m^s \in (0,1)$. Moreover, since $\max\{\Gamma_0(m^s),\Gamma_0(m^s)\}=0$, $m^s=\min\{m_0^s,m_1^s\}$. In Figure 17, I represent the functions $\Gamma_0(m_t)$ and $\Gamma_1(m_t)$ and the determination of m_0^s, m_1^s, and m^s.

Characterization of speculative hyperinflations. For all $m_t < m^s$, $\dot{m}_t < 0$. Hence, there are a continuum of speculative hyperinflation equilibria indexed by $m_0 \in (0, m^s)$ and such that m_t decreases over time with $\lim_{t\to+\infty} m_t = 0$. From (61),

$$a_t \left\{ \begin{array}{c} = \\ > \end{array} \right. 0 \text{ if } \alpha\sigma\chi_2\left[(m_t)^{-\eta}-1\right] \left\{ \begin{array}{c} < \\ > \end{array} \right. \rho - r_a.$$

Equivalently, using the definition of ω_1^s in (65), i.e., $\alpha\sigma\chi_2\left[(\omega_1^s)^{-\eta}-1\right]=\rho-r_a$,

$$a_t \left\{ \begin{array}{c} = \\ > \end{array} \right. 0 \text{ if } m_t \left\{ \begin{array}{c} > \\ < \end{array} \right. \omega_1^s.$$

Since m_t is decreasing over time and approaches zero asymptotically, there is a $T_0 > 0$ such that for all $t < T_0$, $a_t = 0$ and for all $t > T_0$, $a_t > 0$. It follows that the ODE (62) can be rewritten as

$$\begin{aligned} \frac{\dot{m}_t}{m_t} &= \Gamma_0(m_t) \text{ for all } t < T_0 \\ &= \Gamma_1(m_t) \text{ for all } t > T_0. \end{aligned}$$

For all $t > T_0$, the ODE $\dot{m}_t/m_t = \Gamma_1(m_t)$ is identical to (22) where $\alpha\sigma$ has been replaced with $\alpha\sigma\chi_m$ and ρ has been replaced with r_a. Hence, from Corollary 3, the solution is

$$m_t = \left\{(m_1^s)^\eta - e^{\eta(\alpha\sigma\chi_m+r_a+\pi)(t-T_0)}\left[(m_1^s)^\eta-(m_{T_0})^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[T_0,T_0+T_1]}(t),$$

where, from (29) by replacing $\alpha\sigma$ with $\alpha\sigma\chi_m$ and ρ with r_a,

$$T_1 = \frac{\ln\left[1-(m_{T_0}/m_1^s)^\eta\right]^{-1}}{(\alpha\sigma\chi_m+r_a+\pi)\,\eta}.$$

From (61) at equality, $a_t + m_t = \omega_1^s$ for all $t > T_0$.

For all $t < T_0$, the ODE $\dot{m}_t/m_t = \Gamma_0(m_t)$ is identical to (22). Hence, from Corollary 3, the solution is

$$m_t = \left\{(m_0^s)^\eta - e^{\eta(\alpha\sigma+\rho+\pi)t}\left[(m_0^s)^\eta-(m_0)^\eta\right]\right\}^{\frac{1}{\eta}} \mathbb{I}_{[0,T_0]}(t) \text{ for all } t<T_0. \tag{85}$$

Steady states. As shown in the right panel of Figure 17, if $m_1^s < m_0^s$, then $m^s = m_1^s < \omega_1^s$. The condition $m_1^s < \omega_1^s$ is equivalent to $r_a > \chi_m\rho - \chi_2\pi$. From (61), $a^s + m^s = \omega_1^s$ and the steady-state holdings of dollars are

$$a_1^s = \left(\frac{\alpha\sigma\chi_2}{\rho-r_a+\alpha\sigma\chi_2}\right)^{\frac{1}{\eta}} - \left(\frac{\alpha\sigma\chi_m}{\alpha\sigma\chi_m+r_a+\pi}\right)^{\frac{1}{\eta}}. \tag{86}$$

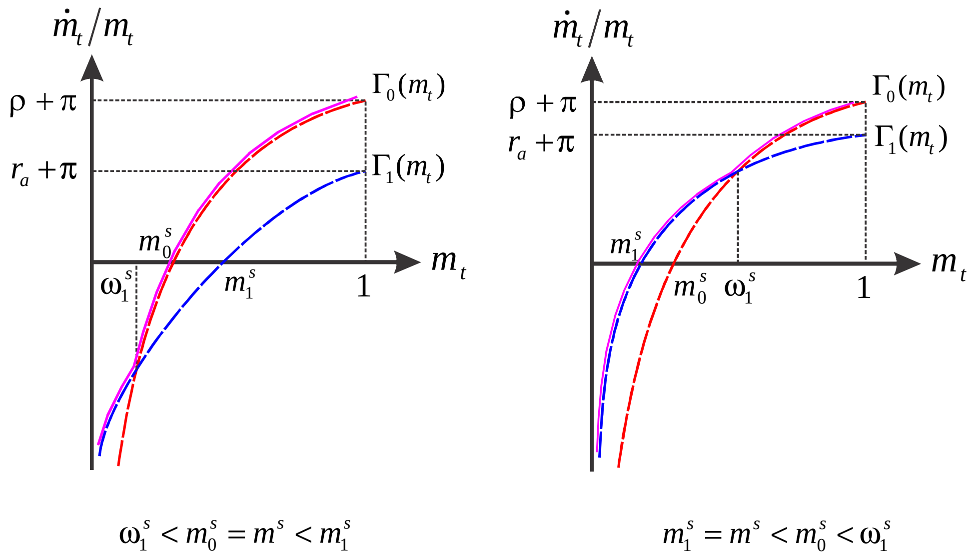

Figure 17: Representation of the functions Γ_0 and Γ_1

For all $m_0 < m^s$, $m_0 < \omega_1^s$. Hence, $a_t > 0$ for all t, i.e., $T_0 = 0$. As shown in the left panel of Figure 17, if $m_0^s < m_1^s$, then $m^s = m_0^s > \omega_1^s$, $a^s = 0$, and $a^s + m^s = m_0^s$. The condition $m_0^s > \omega_1^s$ is equivalent to $r_a < \chi_m \rho - \chi_2 \pi$. For all $m_0 \in (\omega_1^s, m^s)$, $T_0 > 0$.

Determination of T_0. From (62), T_0 is the solution to

$$\alpha\sigma\chi_2 \left[(m_{T_0})^{-\eta} - 1\right] = \rho - r_a,$$

or, equivalently,

$$m_{T_0} = \omega_1^s = \left(\frac{\alpha\sigma\chi_2}{\alpha\sigma\chi_2 + \rho - r_a}\right)^{\frac{1}{\eta}}.$$

Using the expression for m_t given by (85), T_0 is the solution to

$$\left\{(m_0^s)^\eta - e^{\eta(\alpha\sigma+\rho+\pi)T_0}\left[(m_0^s)^\eta - (m_0)^\eta\right]\right\}^{\frac{1}{\eta}} = \omega_1^s.$$

Solving for T_0,

$$T_0 = \frac{1}{\eta(\alpha\sigma + \rho + \pi)} \ln\left[\frac{(m_0^s)^\eta - (\omega_1^s)^\eta}{(m_0^s)^\eta - (m_0)^\eta}\right],$$

which corresponds to (66). If $\omega_1^s < m_0 < m_0^s$, then $T_0 > 0$, as shown in the left panel of Figure 17. ■

Appendix B: Shi-Trejos-Wright model in discrete time

In discrete time, the model can be rewritten as:

$$\begin{aligned} V_{1,t} &= \beta_\Delta \left\{ \alpha_\Delta \sigma (1-M) \left[u(y_{t+1}) + V_{0,t+1} - V_{1,t+1} \right] + V_{1,t+1} \right\} \\ V_{0,t} &= \beta_\Delta \left\{ \alpha_\Delta \sigma M \left(-y_{t+1} + V_{1,t+1} - V_{0,t+1} \right) + V_{0,t+1} \right\}. \end{aligned}$$

The take-it-or-leave-it offer by sellers gives

$$y_{t+1} = V_{1,t+1} - V_{0,t+1}.$$

Hence,

$$y_t = \beta_\Delta \left\{ \alpha_\Delta \sigma (1-M) \left[u(y_{t+1}) - y_{t+1} \right] + y_{t+1} \right\}.$$

We assume $\beta_\Delta = e^{-\rho\Delta}$ and $\alpha_\Delta = 1 - e^{-\alpha\Delta}$.

Suppose $u(y) = \sqrt{y}$. The ODE can be rewritten as:

$$\beta_\Delta \left[1 - \alpha_\Delta \sigma (1-M) \right] y_{t+1} + \beta_\Delta \alpha_\Delta \sigma (1-M) \sqrt{y_{t+1}} - y_t = 0.$$

Adopt the change of variable $x_t = \sqrt{y_t}$. Then,

$$\beta_\Delta \left[1 - \alpha_\Delta \sigma (1-M) \right] x_{t+1}^2 + \beta_\Delta \alpha_\Delta \sigma (1-M) x_{t+1} - x_t^2 = 0.$$

We solve for x_{t+1} as a function of x_t:

$$x_{t+1} = \frac{\sqrt{\left[\beta_\Delta \alpha_\Delta \sigma (1-M) \right]^2 + 4\beta_\Delta \left[1 - \alpha_\Delta \sigma (1-M) \right] x_t^2} - \beta_\Delta \alpha_\Delta \sigma (1-M)}{2\beta_\Delta \left[1 - \alpha_\Delta \sigma (1-M) \right]}.$$

In terms of y_{t+1},

$$y_{t+1} = \left\{ \frac{\sqrt{\left[\beta_\Delta \alpha_\Delta \sigma (1-M) \right]^2 + 4\beta_\Delta \left[1 - \alpha_\Delta \sigma (1-M) \right] y_t} - \beta_\Delta \alpha_\Delta \sigma (1-M)}{2\beta_\Delta \left[1 - \alpha_\Delta \sigma (1-M) \right]} \right\}^2.$$